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Quarter-Symmetric Metric Connection in P-Sasakian Manifold

Dhruwa Narain and Gajendra Nath Tripathi

Department of Mathematics & Statistics
D. D. U. Gorakhpur University,Gorakhpur, India
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Abstract

The object of the present paper is to study properties of curvature tensor of a quarter symmetric metric connection in a P-Sasakian manifold.

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1. Introduction

An $n-$dimensional differentiable manifold M is called an almost para-contact manifold if it admits an almost para-contact structure (F, ξ, η) consisting of a (1, 1) tensor field F, a vector field ξ, and a 1-form η satisfying

$$\overline{(\overline{X})} = X - \eta(X)\,\xi, \tag{1.1}$$

$$\overline{X} = F(X), \tag{1.2}$$

$$\eta(\xi) = 1. \tag{1.3}$$

Let g be the compatible Riemannian metric with (F, ξ, η) that is

$$g(FX, FY) = g(X, Y) - \eta(X)\,\eta(Y) \tag{1.4}$$

or equivalently

$$g(X, FY) = g(FX, Y) \tag{1.5}$$

and

$$g(X, \xi) = \eta(X) \quad \text{for all } X, Y \in TM. \tag{1.6}$$

Then M becomes almost para-contact Riemannian manifold equipped with an almost para-contact Riemannian structure (F, ξ, η, g). An almost para-contact

Riemannian manifold is called a $P-$Sasakian manifold if it satisfies

$$(\nabla_X F)\,Y = -g(X,Y)\,\xi - \eta\,(Y)\,X + 2\eta(X)\,\eta(Y)\,\xi, \qquad X,Y \in TM \tag{1.7}$$

where, ∇ denote the covariant differentiation with respect to g. It follows that

$$(\nabla_X\,\xi) = \overline{X} \tag{1.8}$$

$$(\nabla_X \eta)\,Y = (\nabla_Y\,\eta)\,X = g(X,\overline{Y}), \qquad X \in TM. \tag{1.9}$$

In an $n-$dimensional $P-$Sasakian manifold M, the curvature tensor R, the Ricci tensor S and the Ricci operator Q, satisfy

$$R(X,Y)\,\xi = \eta\,(X)\,Y - \eta\,(Y)\,X \tag{1.10}$$

$$R(\xi,X)Y = \eta\,(Y)X - g(X,Y)\,\xi \tag{1.11}$$

$$R(\xi,X)\,\xi = X - \eta\,(X)\,\xi \tag{1.12}$$

$$S(X,\xi) = -(n-1)\eta\,(X) \tag{1.13}$$

$$Q\xi, = -(n-1)\,\xi, \tag{1.14}$$

$$\eta\,(R(X,Y)U) = g(X,U)\eta\,(Y) - g(Y,U)\eta\,(X) \tag{1.15}$$

$$\eta\,(R(X,Y)\,\xi) = 0 \tag{1.16}$$

$$\eta\,(R(\xi,X)Y) = \eta\,(X)\,\eta\,(Y) - g(X,Y) \tag{1.17}$$

An almost para contact Riemannian manifold M is said to be $\eta-$Einstein [4], if the Ricci operator Q satisfying

$$Q(X) = aX + b\eta(X)\,\xi,$$

where a and b are smooth function on the manifold. In particular if $b = 0$, the M is an Einstein manifold.

Let (M, g) be an n-dimensional Riemannian manifold. Then the projective curvature tensor P and the Weyl Conformal tensor C are defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y] \tag{1.18}$$

$$C(X,Y)U = R(X,Y)U - \frac{1}{(n-2)}\{S(Y,U)X - S(X,U)Y + (g(Y,U)QX - (g(X,U)QY\} + \frac{r}{(n-1)(n-2)}\{g(Y,U)X - g(X,U)Y\} \tag{1.19}$$

for all $X, Y \in TM$ respectively, where r is the scalar curvature of M.

A linear connection $\tilde{\nabla}$ in a Riemannian manifold M is said to be a quarter symmetric connection if its torsion tensor T satisfies

$$T(X,Y) = \eta(Y)\phi(X) - \eta(X)\phi(Y) \tag{1.20}$$

where η is a 1-form and ϕ is a (1, 1) tensor field [2].

A linear connection $\tilde{\nabla}$ is called a metric connection if

$$(\tilde{\nabla}_X g)(Y,Z) = 0. \tag{1.21}$$

A linear connection $\tilde{\nabla}$ satisfying (1.20) and (1.21) is called a quarter symmetric metric connection [3].

2. Curvature Tensor

We consider a linear connection and be a Riemannian connection such that

$$\tilde{\nabla}_X Y = \nabla_X Y + U(X,Y) \tag{2.1}$$

where U is a tensor of type (1, 2), and

$$T(X,Y) = \eta(Y)\overline{X} - \eta(X)\overline{Y} = U(X,Y) - U(Y,X) \tag{2.2}$$

If a connection $\tilde{\nabla}$ is metric connection, i.e.

$$(\tilde{\nabla}_X g)(Y,Z) = 0. \tag{2.3}$$

holds. From (2.2) we have

$$'U(X,Y,Z) + {'U}(X,Z,Y) = 0 \tag{2.4}$$

where $'U(X,Y,Z) = g(U(X,Y),Z)$. Since

$$(\tilde{\nabla}_X g)(Y,Z) = 0 \;\square\; g(\tilde{\nabla}_X Y,Z) + g(Y,\tilde{\nabla}_X Z) = g(\tilde{\nabla}_X Y,Z) + g(Y,\tilde{\nabla}_X Z)$$
$$\square\; g(U(X,Y),Z) + g(Y,U(X,Z)) = 0$$
$$\square\; {'U}(X,Y,Z) + {'U}(X,Z,Y) = 0$$

and also

$$U(X,Y) = \frac{1}{2}[T(X,Y) + T'(X,Y) + T'(Y,X)] \tag{2.5}$$

where

$$g(T'(Y,X),Z) = g(T(Z,X),Y) \tag{2.6}$$

Assume that the torsion tensor $T(X,Y)$ of the linear connection is of the form

$$T(X,Y) = \eta(Y)\overline{X} - \eta(X)\overline{Y}. \tag{2.7}$$

From (2.6) and (2.7), we have

$$T(X,Y) = \eta(X)\overline{Y} - {'F}(X,Y) \tag{2.8}$$

where ${'F}(X,Y) = g(X,Y)$, η is a 1-form and ξ, is the associated vector field.

From (2.5), (2.7) and (2.8), we get

$$U(X,Y) = \eta(Y)\overline{X} - {'F}(X,Y)\,\xi, \tag{2.9}$$

From (2.1) and (2.9), we get

$$\tilde{\nabla}_X\, Y = \nabla_X Y + \eta(Y)\overline{X} - {'F}(X,Y)\,\xi, \tag{2.10}$$

Hence a quarter symmetric metric connection $\tilde{\nabla}$ in a $P-$Sasakian manifold is given by (2.10).

If R and $\tilde{R}$ be the curvature tensors of the connection ∇ and $\tilde{\nabla}$ respectively. Then we have

$$R(X,Y)Z = \nabla_X\,\nabla_Y\,Z - \nabla_Y\,\nabla_X\,Z - \nabla_{[X,Y]}\,Z \tag{2.11}$$

$$\tilde{R}(X,Y)\,Z = \nabla_X\,\nabla_Y\,Z - \nabla_Y\,\nabla_X\,Z - \nabla_{[X,Y]}\,Z. \tag{2.12}$$

Using (2.10) and (2.11) in (2.12), we have

$$\tilde{R}(X,Y)Z = R(X,Y)Z + 3\,{'F}(X,Z)\overline{Y} - 3\,{'F}(Y,Z)\overline{X} + [(\nabla_X\,F)(Y) \\ (\nabla_Y\,F)(X)]\,\eta(Z) - [(\nabla_X\,{'F})(Y,Z) - (\nabla_Y\,{'F}(X,Z))]\,\xi. \tag{2.13}$$

From (1.7) and (2.13), we have

$$\tilde{R}(X,Y)Z = R(X,Y)Z + 3\,{'F}(X,Z)\overline{Y} - 3\,{'F}(Y,Z)\overline{X} + [\eta(X)Y \\ -\eta(Y)X]\eta(Z) - [\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]\,\xi. \tag{2.14}$$

From (2.14), we have

$$\begin{aligned} {'\tilde{R}}(X,Y,Z,U) = {}& {'R}(X,Y,Z,U) + 3\,{'F}(X,Z)\,{'F}(Y,U) \\ & -3\,{'F}(Y,Z)\,{'F}(X,U) + \eta(X)\eta(Z)g(Y,U) \\ & -\eta(Y)\eta(Z)g(X,U) - [\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]\eta(U). \end{aligned} \tag{2.15}$$

A relation between the curvature tensor of M with respect to the quarter-symmetric connection $\tilde{\nabla}$ and the Riemannian connection, ∇ is given by the equation (2.15).

$$\tilde{S}(Y,Z) = S(Y,Z) + \eta(Y)\eta(Z) - n\eta(Y)\eta(Z) - g(Y,Z) + \eta(Y)\eta(Z)$$

$$\tilde{S}(Y,Z) = S(Y,Z) - (n-2)\eta(Y)\eta(Z) - g(Y,Z) \tag{2.16}$$

Contracting (2.16) with respect to z, we get

$$\tilde{r} = r - (n-2) \tag{2.17}$$

where $\tilde{r}$ and r are the scalar curvatures of the connection $\tilde{\nabla}$ and ∇ respectively.

Theorem 1. For a $P-$Sasakian manifold M with quarter symmetric metric connection $\tilde{\nabla}$, we have

(a) $\tilde{R}(X,Y)\,Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0$

(b) $'\tilde{R}(X,Y,Z,U) + {'\tilde{R}}(X,Y,U,Z) = 0$

(c) $'\tilde{R}(X,Y,Z,U) -' \tilde{R}(Z,U,X,Y) = 0$

(d) $'\tilde{R}(X,Y,Z,\xi) = 2'\tilde{R}(X,Y,Z,\xi) = 0$

(e) $\tilde{S}(X,\xi) = 2S(X,\xi)$.

Proof: (a) of the theorem we have from (2.14). From (2.15) we have (b) and (c) of the theorem. Putting $U = \xi$ in (2.15) we have (d) of the theorem. Putting $Y = Z = e_i$ in (d) and taking summation over i, we get (e) of the theorem.

Theorem 2. In a $P-$Sasakian manifold in the Ricci tensor of the quarter symmetric metric connection is symmetric.

Proof: The proof of the theorem obviously follows from (2.16).

The Projective curvature tensor $'P$ of type (0, 4) of M with respect to Riemannian connection is given by

(2.18)
$$'P(X,Y,Z,U) = {'R}(X,Y,Z,U) - \frac{1}{n-1}[S(Y,Z)\,g(X,U) - S(X,Z)g(Y,U)].$$

Analogous to this definition, we define Projective curvature tensor $'P$ of M with respect with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ by

(2.19)
$$'\tilde{P}(X,Y,Z,U) = {'\tilde{R}}(X,Y,Z,U) - \frac{1}{n-1}[\tilde{S}(Y,Z)g(X,U) - \tilde{S}(X,Z)g(Y,U)].$$

From (2.15), (2.16), (2.18) and (2.19), we have

(2.20)
$$'\tilde{P}(X,Y,Z,U) = {'P}(X,Y,Z,U) + 3'F(X,Z)\,'F(Y,U) - 3'F(Y,Z)'F(X,U)$$
$$+\frac{1}{n-1}[g(\overline{Y},\overline{Z})g(X,U) - g(\overline{X},\overline{Z})g(Y,U)] + [\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]\,\eta(U).$$

Hence we can state the following theorem.

Theorem 3. In a $P-$Sasakian manifold the projective curvature tensor $\tilde{P}$ of a quarter-symmetric metric connection $\tilde{\nabla}$ satisfying

(i) $$'\tilde{P}(X,Y,Z,U) = {}'P(X,Y,Z,U) + 3\,'F(X,Z)\,'F(Y,U) - 3\,'F(Y,Z)\,'F(X,U) + \frac{1}{n-1}[g(\overline{Y},\overline{Z})g(X,U) - g(\overline{X},\overline{Z})\,g(Y,U)] + [\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]\,\eta(U)$$

(ii) $'\tilde{P}(X,Y,Z,U) +' \tilde{P}(Y,Z,X,U) +' \tilde{P}(Z,X,Y,U) = 0$

(iii) $'\tilde{P}(X,Y,Z,\xi) = \left(\frac{1}{n-1} - 1\right)[\eta(X)g(Y,Z) - \eta(Y)g(Z,X)]$.

Weyl conformal curvature tensor $'C$ of type (0, 4) of M^n with respect to the Riemannian connection is given by

$$\begin{aligned}'C(X,Y,Z,U) = {}'R(X,Y,Z,U) &- \frac{1}{n-2}[S(Y,Z)g(X,U) - S(X,Z) \\ &g(Y,U) + S(X,U)g(Y,Z) - S(Y,U)g(X,Z)] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)].\end{aligned} \tag{2.21}$$

Analogous to this definition, we define conformal curvature tensor of M^n with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ by

$$\begin{aligned}'C(X,Y,Z,U) = {}'R(X,Y,Z,U) &- \frac{1}{n-2}[\tilde{S}(Y,Z)g(X,U) - \tilde{S}(X,Z) \\ &g(Y,U) + \tilde{S}(X,U)g(Y,Z) - \tilde{S}(Y,U)g(X,Z)] \\ &+ \frac{\tilde{r}}{(n-1)(n-2)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)].\end{aligned} \tag{2.22}$$

From (2.15), (2.16), (2.17), (2.21) and (2.22), we have

$$\begin{aligned}'\tilde{C}(X,Y,Z,U) = {}'C(X,Y,Z,U) &+ 3'F(X,Z)'F(Y,U) \\ &- 3'F(Y,Z)'F(X,U)\end{aligned} \tag{2.23}$$

$$'\tilde{C}(X,Y,Z,\xi) = {}'C(X,Y,Z,\xi). \tag{2.24}$$

Hence we can state the following theorem:

Theorem 4. In a $P-$Sasakian manifold the Weyl conformal curvature tensor $\tilde{C}$ of a quarter symmetric metric connection $\tilde{\nabla}$ satisfying

(i) $'\tilde{C}(X,Y,Z,U) = {}'C(X,Y,Z,U) + 3\,'F(X,Z)\,'F(Y,U) - 3\,'F(Y,Z)\,'F(X,U)$

(ii) $'\tilde{C}(X,Y,Z,U) + {}'\tilde{C}(Y,Z,X,U) + {}'\tilde{C}(Z,X,Y,U) = 0$

(iii) $'\tilde{C}(X,Y,Z,\xi) = {}'C(X,Y,Z,\xi)$.

3. Einstein Manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ in a P-Sasakian Manifold

A Riemannian manifold M^n is called an Einstein manifold with respect to Riemannian connection if

$$S(X,Y) = \frac{r}{n}g(X,Y) \tag{3.1}$$

Analogous to this definition, we define Einstein manifold with respect to quarter-symmetric metric connection $\tilde{\nabla}$ by

$$\tilde{S}(X,Y) = \frac{\tilde{r}}{n}\,g(X,Y). \tag{3.2}$$

From (2.16), (2.17), (3.1) and (3.2), we have

(3.3)

$$\tilde{S}(X,Y) - \frac{\tilde{r}}{n}\,g(X,Y) = S(X,Y) - (n-2)\eta(X)\eta(Y) - g(X,Y) - \frac{r-(n-2)}{n}\,g(X,Y).$$

If

$$g(X,Y) = n\Big(\frac{n}{2}-1\Big)\eta(X)\eta(Y). \tag{3.4}$$

Then from (3.3), we get

$$\tilde{S}(X,Y) - \frac{\tilde{r}}{n}\,g(X,Y) = S(X,Y) - \frac{r}{n}g(X,Y). \tag{3.5}$$

Hence we can state the following theorem :

Theorem 5. In a $P-$Sasakian manifold M^n with quarter-symmetric metric connection $\tilde{\nabla}$ if the relation $g(X,Y) = n\Big(\frac{n}{2}-1\Big)\eta(X)\eta(Y)$ holds, then the manifold is an Einstein manifold for Riemannian connection if and only if it is Einstein manifold for the connection $\tilde{\nabla}$.

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