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On Four Dimensional Finsler Space Satisfying T-Conditions

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Abstract

The purpose of the present paper is to consider the four dimensional Finsler spaces with $T_{hijk} = 0$ and generalize the idea of Landsberg angle to four dimensional Finsler spaces. The properties of a Finsler space satisfying $T-$condition has been studied in a three dimensional Finsler space by various authors ([2], [3], [4], [8], [10]). But from the relativistic point of view the importance of four dimensional Finsler space is not negligible. In relativity the fourth coordinate is taken as time, from this point of view we discuss the properties of four dimensional Finsler space satisfying $T-$condition. The results which are reducible to the three dimensional case also.

1. Introduction

H. Kawaguchi and M. Matsumoto have introduced the T-tensor in a Finsler space independently ([6], [5]). It is indicatrised tensor and studied by several authors ([1], [2], [3], [4], [8]). The vanishing of $T-$tensor is called $T-$condition. Hashiguchi [1] noticed the importance of $T-$tensor from the stand point of Landsberg spaces. It has been proved by him that a necessary and sufficient condition for a Landsberg space to be conformally invariant is that it satisfy $T-$condition.

The Landsberg angle θ was introduced by Landsberg in 1908. The coordinate system (L, θ) in a tangent plane M_x is regarded as a generalization of the polar coordinate system (r, θ) of a Euclidean plane. M. Matsumoto [9] gave the idea of Landsberg angle in two and three dimensional Finsler space.

In this paper we have considered four dimensional Finsler space with $T_{hijk} = 0$, and generalized the idea of Landsberg angle to four dimensional Finsler spaces.

Let M^4 be four dimensional Finsler space endowed with a fundamental function $L = L(x, y)$, where $x = (x^i)$ is a point and $y = (y^i)$ is a supporting element of M^4. The metric tensor g_{ij} and (h) hv-torsion tensor C_{ijk} of M^4 is given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}, \qquad C_{ijk} = \frac{1}{2}\frac{\partial^3 L^2}{\partial y^i \partial y^j \partial y^k}. \tag{1.1}$$

If $[g^{ij}]$ denote the inverse matrix of $[g_{ij}]$ then, we have $g_{ij}g^{jk} = \delta_i^k$. The $T-$tensor T_{ijkl} is defined as

$$T_{hijk} = LC_{hij}\,|_k + l_h C_{ijk} + l_i C_{hjk} + l_j C_{hik} + l_k C_{hij}\ , \tag{1.2}$$

where $l_i = L^{-1}\,g_{ir}y^r$ and '$|$' denotes the $v-$covariant derivative with respect to Cartan connection $C\Gamma$ of M^4. For instance the $v-$covariant derivative of a tensor field $T^i_j(x, y)$ is defined by

$$T^i_j\Big|_k = \dot\partial_k T^i_j + T^r_j C^i_{rk} - T^i_r C^r_{jk}\ , \tag{1.3}$$

where $\dot\partial_k = \frac{\partial}{\partial y^i}$, $\partial_k = \frac{\partial}{\partial x^k}$.

2. Scalar components in Miron frame

Let M^4 be a four dimensional Finsler space with the fundamental function $L(x, y)$. The frame $\{e^i_{\alpha)}\}$, $\alpha = 1, 2, 3, 4$ is called the Miron's frame of M^4, where $e^i_{1)} = l^i = y^i/L$ is the normalized supporting element, $e^i_{2)} = m^i = C^i/C$ is the normalized torsion vector, $e^i_{3)} = n^i$, $e^i_{4)} = p^i$ are constructed by $g_{ij}e^i_{\alpha)}e^j_{\beta)} = \delta_{\alpha\beta}$. Here C is the length of torsion vector $C_i = C_{ijk}g^{jk}$. The Greek letters $\alpha, \beta, \gamma, \delta$ varies from 1 to 4. Summation convention is applied for both the Greek and Latin indices.

In Miron's frame an arbitrary tensor field can be expressed by scalar components along the unit vectors $e^i_{\alpha)}$, $\alpha = 1, 2, 3, 4$. For instance, let T^i_j be a tensor field of type (1, 1), then the scalar components $T_{\alpha\beta}$ of T^i_j are defined by $T_{\alpha\beta} = T^i_j e_{\alpha)i} e^j_{\beta)}$ and the components T^i_j are expressed as $T^i_j = T_{\alpha\beta}e^i_{\alpha)}e_{\beta)j}$. From the equation $g_{ij}e^i_{\alpha)}\,e^j_{\beta)} = \delta_{\alpha\beta}$, we have

$$g_{ij} = l_i l_j + m_i m_j + n_i n_j + p_i p_j. \tag{2.1}$$

The C-tensor $C_{ijk} = \frac{1}{2}\frac{\partial g_{jk}}{\partial y^i}$ satisfies $C_{ijk}l^k = 0$ and is symmetric in i, j, k therefore if $C_{\alpha\beta\gamma}$ be the scalar components of LC_{ijk}, i.e. if

$$LC_{ijk} = C_{\alpha\beta\gamma}e_{\alpha)i}e_{\beta)j}e_{\gamma)k}, \tag{2.2}$$

then, we have [10]

$$\begin{aligned} LC_{ijk} &= C_{222}m_im_jm_k + C_{333}n_in_jn_k + C_{444}p_ip_jp_k + C_{233}\pi_{(ijk)}(m_in_jn_k) \\ &+C_{244}\pi_{(ijk)}(m_ip_jp_k) + C_{344}\pi_{(ijk)}(n_ip_jp_k) + C_{322}\pi_{(ijk)}(m_im_jn_k) \\ &+C_{433}\pi_{(ijk)}(n_in_jp_k) + C_{422}\pi_{(ijk)}(m_im_jp_k) + C_{234}\pi_{(ijk)}\{m_i(n_jp_k + n_kp_j)\}, \end{aligned} \tag{2.3}$$

where $\pi_{(ijk)}$ denote the cyclic permutation of indices i, j, k and summation. For instance

$$\pi_{(ijk)}(A_iB_jC_k) = A_iB_jC_k + B_iC_jA_k + C_iA_jB_k.$$

Contracting (2.2) with g^{jk}, we get $LCm_i = C_{\alpha\beta\beta}e_{\alpha)i}$. Thus if we put

$$\begin{aligned} &C_{222} = H, \quad C_{233} = I, \quad C_{244} = K, \quad C_{333} = J, \\ &C_{344} = J', \quad C_{444} = H', \quad C_{433} = I', \quad C_{234} = K', \end{aligned} \tag{2.4}$$

then we have

$$H + I + K = LC, \qquad C_{322} = -(J + J'), \qquad C_{422} = -(H' + I'). \tag{2.5}$$

The eight scalars $H, I, J, K, H', I', J', K'$ are called the main scalars of a four dimensional Finsler space.

The $v-$covariant derivative of the frame field $e_{\alpha)i}$ is given by

$$Le_{\alpha)i}|_j = V_{\alpha)\beta\gamma}e_{\beta)i}e_{\gamma)j}, \tag{2.6}$$

where $V_{\alpha)\beta\gamma}$, γ being fixed are given by

$$V_{\alpha)\beta\gamma} = \begin{bmatrix} 0 & \delta_{2\gamma} & \delta_{3\gamma} & \delta_{4\gamma} \\ \delta_{2\gamma} & 0 & u_\gamma & v_\gamma \\ \delta_{3\gamma} & -u_\gamma & 0 & w_\gamma \\ \delta_{4\gamma} & -v_\gamma & -w_\gamma & 0 \end{bmatrix} \quad \text{and} \quad \begin{array}{l} V_{2)3\gamma} = -V_{3)2\gamma} = u_\gamma \\ V_{2)4\gamma} = -V_{4)2\gamma} = v_\gamma \\ V_{3)4\gamma} = -V_{4)3\gamma} = w_\gamma \end{array} \tag{2.7}$$

Thus, in a four dimensional Finsler space there exists three $v-$connection vectors u_i, v_i, w_i whose scalar components with respect to the frame $\{e^i_{\alpha)}\}$ are u, v, w, i.e.

$$u_i = ue_{\gamma)i}, \qquad v_i = ve_{\gamma)i} \qquad w_i = we_{\gamma)i}. \tag{2.8}$$

In view of equations (2.8), the equation (2.6) may be explicitly written as

$$(2.9)\qquad \begin{array}{ll} Ll_i|_j = m_i m_j + n_i n_j + p_i p_j & Lm_i|_j = -\,l_i m_j + n_i u_j + p_i v_j, \\ Ln_i|_j = -l_i n_j - m_i u_j + p_i w_j, & Lp_i|_j = -\,l_i p_j - m_i v_j - n_i w_j. \end{array}$$

Since m_i, n_i, p_i are homogeneous functions of degree zero in y_i, we have

$$Lm_i|_j l^j = Ln_i|_j l^j = Lp_i|_j l^j = 0,$$

which in view of equations (2.8) and (2.9) gives $u_1 = 0$, $v_1 = 0$, $w_1 = 0$. Therefore

Lemma (2.1). The first scalar components u_1, v_1, w_1 of the $v-$connection vectors u_i, v_i, w_i vanishes identically, that is u_i, v_i, w_i are orthogonal to l^i.

3. Four-dimensional Finsler space satisfying the T-condition

The scalar derivative of the adopted components $T_{\alpha\beta}$ of T^i_j is defined as [9]

$$(3.1)\qquad T_{\alpha\beta;\gamma} = L(\partial_k T_{\alpha\beta}) e_{\gamma)}{}^k + T_{\mu\beta} V_{\mu)\alpha\gamma} + T_{\alpha\mu} V_{\mu)\beta\gamma},$$

Thus $T_{\alpha\beta;\gamma}$ are adopted components of $LT^i_j\,|_k$, i.e.

$$(3.2)\qquad LT^i_j\,|_k = T; e^i_{\alpha)} e_{\beta)j} e_{\gamma)k}.$$

If the tensor field T^i_j is positively homogeneous of degree zero in y^i, $T_{\alpha\beta}$ is also positively homogeneous of degree zero in y^i, so equation (3.1) gives

$$T_{\alpha\beta;1} = T_{\mu\beta} V_{\mu)\alpha 1} + T_{\alpha\mu} V_{\mu)\beta 1},$$

which in view of (2.7) and lemma (2.1) gives $T_{\alpha\beta;1} = 0$. Therefore we have the following:

Proposition (3.1). If the tensor field T^i_j is positively homogeneous of degree zero in y^i, then $T_{\alpha\beta;1} = 0$.

Now, let T^i_j be positively homogenous of degree r in y^i and $T_\alpha\beta$ be the scalar components of $L^{-r}T^i_j$, then $L(L^{-r}T^i_j)\,|_k = T_{\alpha\beta;\gamma} e^i_{\alpha)} e_{\beta)j} e_{\gamma)k} = L^{-r+1} T^i_j\,|_k - rL^{-r}$ $T^i_j e_{1)k}$, which implies

$$(3.3)\qquad L^{-r+1} T^i_j\,|_k = (T_{\alpha\beta;\gamma} + rT_{\alpha\beta}\delta_{1\gamma}) e^i_{\alpha)} e_{\beta)j} e_{\gamma)k}.$$

Hence we have

Proposition (3.2). If the tensor field T^i_j is positively homogeneous of degree r in y^i and $T_{\alpha\beta}$ be the scalar components of $L^{-r}T^i_j$, then the scalar components of $L^{-r+1}T^i_j\,|_k$ are given by $T_{\alpha\beta;\gamma} + rT_{\alpha\beta}\delta_{1\gamma}$.

Definition (3.1). The Finsler space M^4 is said to satisfy the T-condition if the $T-$tensor T_{hijk} of M^4 vanishes identically.

The C-tensor C_{ijk} is positively homogeneous of degree -1 in y^i, therefore from proposition (3.2) the scalar components of $L^2C_{ijk}\,|_h$ are given by

(3.4) $$L^2C_{ijk}\,|_h = (C_{\alpha\beta\gamma;\delta} - C_{\alpha\beta\gamma}\delta_{1\delta})e_{\alpha)i}e_{\beta)j}e_{\gamma)k}e_{\delta)h},$$

And the scalar components $T\alpha\beta\gamma\delta$ of LT_{hijk} are given by

(3.5) $$T_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma;\delta} + C_{\beta\gamma\delta}\delta_{1\alpha} + C_{\alpha\gamma\delta}\delta_{1\beta} + C_{\alpha\beta\delta}\delta_{1\gamma} + C_{\alpha\beta\gamma}\delta_{1\delta}.$$

We know that the $T-$tensor is indicatrized tensor and is symmetric in all indices, therefore $T_{hijk}\,l^k = 0$ i.e. $T_{\alpha\beta\gamma 1} = 0$. Therefore, the surviving scalar components of LT_{hijk} are given by

(3.6) $$T_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma;\delta} \qquad \alpha,\beta,\gamma,\delta = 2,3,4.$$

Since $C_{hij}\,|_k = C_{hik}\,|_j$, from (3.4) we have

(3.7) $$C_{\alpha\beta\gamma;\delta} - C_{\alpha\beta\gamma}\delta_{1\delta} = C_{\alpha\beta\delta;\gamma} - C_{\alpha\beta\delta}\delta_{1\gamma}.$$

In case of $(\gamma,\delta) = (1,2)$, $(1,3)$ and $(1,4)$ the above relation is trivial and when (γ,δ) = (2, 3), (2, 4), (3, 4), we get

(3.8) $$C_{\alpha\beta 3;2} = C_{\alpha\beta 2;3}, \qquad C_{\alpha\beta 4;2} = C_{\alpha\beta 2;4}, \qquad C_{\alpha\beta 4;3} = C_{\alpha\beta 3;4}.$$

These equations are trivial for $\alpha,\beta = 1$. Consequently, we put (α,β) = (2, 2), (2, 3), (2, 4), (3, 3), (3.4), (4, 4) in equation (3.8). For instance $C_{223;2} = C_{222;3}$ etc. In view of (2.4) and (2.7), this equation is explicitly written as

$$(\dot{\partial}_i C_{223})e^i_{2)} + 2C_{\mu 23}V_{\mu)22} + C_{\mu 22}V_{\mu)32} = (\dot{\partial}_i C_{222})e^i_{3)} + 3C_{\mu 22}V_{\mu)23},$$

Or

(3.9)(a) $$-(J+J')_{;2} + (H-2I)\,u_2 - 2K'v_2 + (H'+I')\,w_2$$
$$= H_{;3} + 3(J+J')u_3 + 3(H'+I')\,v_3.$$

Similarly, from (2.4), (2.7) and (3.8), we get

(3.9)(b) $$I_{;2} - (3J+2J')u_2 - I'v_2 - 2K'w_2$$

$$= -(J+J')_{;3} + (H-2I)\,u_3 - 2K'v_3 + (H'+I')\,w_3,$$

(c) $$K'_{;2} - (H'+2I')\,u_2 - (J+2J')\,v_2 + (I-K)\,w_2$$
$$= (H'+I')_{;3} - 2K'u_3 + (H-2K)\,v_3 - (J+J')\,w_3,$$
$$= (J+J')_{;4} + (H-2I)\,u_4 - 2K'\,v_4 + (H'+I')\,w_4,$$

(d) $$J_{;2} + 3Iu_2 - 3I'\,w_2 = I_{;3} - (3J+2J')\,u_3 - I'v_3 - 2K'w_3,$$

(e) $$I'_{;2} + 2K'u_2 + Iv_2 + (J-2J')w_2$$
$$= K'_{;3} - (H'+2I')u_3 - (J+2J')v_3 + (I-K)w_3$$
$$= I_{;4} - (3J+2J')u_4 - I'v_4 - 2K'w_4,$$

(f) $$J'_{;2} + Ku_2 + 2K'v_2 + (2I'-H')w_2 = K_{;3} - J'u_3 - (3H'+2I')v_3 + 2K'w_3$$
$$= K'_{;4} - (H'+2I')u_4 - (J+2J')v_4 + (I-K)w_4,$$

(g) $$-(H'+I')_{;2} - 2K'u_2 + (H-2K)v_2 - (J+J')w_2$$
$$= H_{;4} + 3(J+J')u_4 + 3(H'+I')v_4,$$

(h) $$K_{;2} - J'u_2 - (3H'+2I')v_2 + 2K'\,w_2$$
$$= -(H'+I')_{;4} - 2K'u_4 + (H-2K)v_4 - (J+J')w_4,$$

(i) $$H'_{;2} + 3Kv_2 + 3J'w_2 = K_{;4} - J'u_4 - (3H'+2I')v_4 + 2K'w_4,$$

(j) $$I'_{;3} + 2K'u_3 + Iv_3 + (J-2J')w_3 = J_{;4} + 3Iu_4 - 3I'w_4,$$

(k) $$J'_{;3} + Ku_3 + 2K'v_3 + (2I'-H')w_3 = I'_{;4} + 2K'u_4 + Iv_4 + (J-2J')w_4,$$

(l) $$H'_{;3} + 3Kv_3 + 3J'w_3 = J'_{;4} + Ku_4 + 2K'v_4 + (2I'-H')w_4.$$

Since T_{hijk} is symmetric in all indices and $T_{1\beta\gamma\delta} = 0$, $\beta, \gamma, \delta = 2,3,4$, therefore, the surviving independent components are fifteen and they are

$$T_{2222},\quad T_{2223},\quad T_{2224},\quad T_{2234},\quad T_{2244},\quad T_{2233},$$
$$T_{2333},\quad T_{2334},\quad T_{2344},\quad T_{2444},\quad T_{3333},\quad T_{3334},$$
$$T_{3344},\quad T_{3444},\quad T_{4444}.$$

In view of (2.4), (2.7), (3.6) and (3.9) these scalar components are explicitly written as

$$T_{2222} = H_{;2} + 3(J+J')u_2 + 3(H'+I')v_2,$$
$$T_{2223} = H_{;3} + 3(J+J')u_3 + 3(H'+I')v_3$$

$$= -(J+J')_{;2} + (H-2I)u_2 - 2K'v_2 + (H'+I')w_2,$$

$$T_{2224} = H_{;4} + 3(J+J')u_4 + 3(H'+I')v_4$$

$$= -(H'+I')_{;2} - 2K'u_2 + (H-2K)v_2 - (J+J')w_2,$$

$$T_{2234} = -(J+J')_{;4} + (H-2I)u_4 - 2K'v_4 + (H'+I')w_4$$

$$= -(H'+I')_{;3} - 2K'u_3 + (H-2K)v_3 - (J+J')w_3$$

$$= K'_{;2} - (H'+2I')u_2 - (J+2J')v_2 + (I-K)w_2,$$

$$T_{2244} = -(H'+I')_{;4} - 2K'u_4 + (H-2K)v_4 - (J+J')w_4$$

$$= K_{;2} - J'u_2 - (3H'+2I')v_2 + 2K'w_2,$$

$$T_{2233} = -(J+J')_{;3} + (H-2I)u_3 - 2K'v_3 + (H'+I')w_3$$

$$= I_{;2} - (3J+2J')u_2 - I'v_2 - 2K'w_2,$$

$$T_{2333} = I_{;3} - (3J+2J')u_3 - I'v_3 - 2K'w_3 = J_{;2} + 3Iu_2 - 3I'w_2,$$

$$T_{2334} = I_{;4} - (3J+2J')u_4 - I'v_4 - 2K'w_4$$

$$= K'_{;3} - (H'+2I')u_3 - (J+2J')v_3 + (I-K)w_3$$

$$= I'_{;2} + 2K'u_2 + Iv_2 + (J-2J')w_2,$$

$$T_{2344} = K'_{;4} - (H'+2I')u_4 - (J+2J')v_4 + (I-K)w_4$$

$$= K_{;3} - J'u_3 - (3H'+2I')v_3 + 2K'w_3$$

$$= J'_{;2} + Ku_2 + 2K'v_2 + (2I'-H')w_2,$$

$$T_{2444} = K_{;4} - J'u_4 - (3H'+2I')v_4 + 2K'w_4 = H'_{;2} + 3Kv_2 + 3J'w_2,$$

$$T_{3333} = J_{;3} + 3Iu_3 - 3I'w_3,$$

$$T_{3334} = J_{;4} + 3Iu_4 - 3I'w_4 = I'_{;3} + 2K'u_3 + Iv_3 + (J-2J')w_3,$$

$$T_{3344} = I'_{;4} + 2K'u_4 + Iv_4 + (J-2J')w_4 = J'_{;3} + Ku_3 + 2K'v_3 + (2I'-H')w_3,$$

$$T_{3444} = J'_{;4} + Ku_4 + 2K'v_4 + (2I'-H')w_4 = H'_{;3} + 3Kv_3 + 3J'w_3,$$

$$T_{4444} = H'_{;4} + 3Kv_4 + 3J'w_4.$$

Now, we consider four dimensional Finsler space with vanishing $T-$tensor, then all the scalar components $T_{\alpha\beta\gamma\delta} = 0,\ \alpha,\beta,\gamma,\delta = 1,2,3,4$. Thus $T_{2234} = T_{3334} = T_{3444} = 0$ gives

$$-(J+J')_{;4} + (H-2I)u_4 - 2K'v_4 + (H'+I')w_4 = 0, \tag{3.10}$$

$$J;_4 + 3Iu_4 - 3I'w_4 = 0, \tag{3.11}$$

$$J';_4 + Ku_4 + 2K'v_4 + (2I' - H')w_4 = 0. \tag{3.12}$$

Adding (3.11), (3.12) and (3.10), we get

$$(H + I + K)u_4 = 0. \tag{3.13}$$

Using (2.5) in (3.13) we get $LCu_4 = 0$. Since $LC \neq 0$, we have $u_4 = 0$.

Similarly, from $T_{2223} = T_{2233} = T_{2333} = T_{2344} = T_{3333} = T_{3344} = 0$, we get $u_3 = u_4 = 0$. Thus $u_\alpha = 0$ for $\alpha = 1, 2, 3, 4$ which implies $u_i = 0$.

Again $T_{2244} = T_{3344} = T_{4444} = 0$ gives

$$-(H' + I');_4 - 2K'u_4 + (H - 2K)v_4 - (J + J')w_4 = 0, \tag{3.14}$$

$$I';_4 + 2K'u_4 + Iv_4 + (J - 2J')w_4 = 0, \tag{3.15}$$

$$H';_4 + 3Kv_4 + 3J'w_4 = 0. \tag{3.16}$$

Adding (3.14), (3.15) and (3.16) we get $(H + I + K)v_4 = 0$ which implies $v_4 = 0$.

Similarly, $T_{2224} = T_{2234} = T_{2334} = T_{2444} = T_{3334} = T_{3444} = 0$ give $v_2 = 0 = v_3$. Therefore $v_\alpha = 0$ for $\alpha = 1, 2, 3, 4$ which implies $v_i = 0$. Putting $u_2 = 0, u_3 = 0, v_2 = 0$, $v_3 = 0, u_4 = 0, v_4 = 0$ in $T_{2222} = 0, T_{2223} = 0$ and $T_{2224} = 0$ we get, $H;_2 = 0$, $H;_3 = 0$ and $H;_4 = 0$. Thus $H;_\alpha = 0$, for $\alpha = 2, 3, 4$. Putting $u_2 = 0$, $v_2 = 0$ in $T_{2234} = 0$, $u_3 = 0$, $v_3 = 0$ in $T_{2344} = 0$ and $u_4 = 0$, $v_4 = 0$ in $T_{2444} = 0$, we get

$$K';_2 + (I - K)w_2 = 0, \qquad K';_3 + (I - K)w_3, \qquad K;_4 + 2K'w_4 = 0. \tag{3.17}$$

We consider two cases.

Case 1. If $I \neq K$ and $K';_\alpha = 0$ for $\alpha = 2, 3, 4$, then from (3..17) we get $w_\alpha = 0$ for $\alpha = 2, 3, 4$ i.e. $w_i = 0$. Hence $T_{\alpha\beta\gamma\delta} = 0$ gives $H;_\alpha = I;_\alpha = J;_\alpha = K;_\alpha = H';_\alpha = I';_\alpha = J';_\alpha = 0$ for $\alpha = 2, 3, 4$. Since the main scalars H, I, J, K, H', I', J' are positively homogeneous of degree one in y^i, we have $H;_\alpha = I;_\alpha = J;_\alpha = K;_\alpha = H';_\alpha = I';_\alpha = J';_\alpha = 0$ for $\alpha = 1$. Hence the main scalars H, I, J, K, H', I', J' does not depend on y^i. Therefore we have the following:

Theorem (3.1). If main scalar K' is independent of directional arguments y^i, and $I \neq K$, the $T-$condition for a non-Riemannian Finsler space of four dimension is equivalent to the fact that the $v-$connection vectors u_i, v_i, and w_i

vanishes identically and the remaining seven main scalars $H, I, J, K,\ H', I', J'$ are also functions of position alone.

Case 2. If $I = K$ then equation (3.17) gives $K';_\alpha = 0$ for $\alpha = 2, 3, 4$. Also $u_i = 0$, $v_i = 0$ gives $H;_\alpha = 0$. Putting these values in $T_{2233} = 0$, $T_{2244} = 0$, $T_{2333} = 0$, $T_{2344} = 0$, $T_{2334} = 0$, and $T_{2444} = 0$, we get

$$
\begin{aligned}
&I;_2 - 2K'w_2 = 0, \qquad K;_2 + 2K'w_2 = 0\\
&I;_3 - 2K'w_3 = 0, \qquad K;_3 + 2K'w_3 = 0,\\
&I;_4 - 2K'w_4 = 0, \qquad K;_4 + 2K'w_4 = 0.
\end{aligned} \tag{3.17}
$$

These equations gives $I;_\alpha + K;_\alpha = 0$ for $\alpha = 2, 3, 4$. Since $I = K$, we have $I;_\alpha = K;_\alpha = 0$ for $\alpha = 2, 3, 4$. Putting these values in (3.17) we get $w_2 = w_3 = w_4 = 0$, provided $K' \neq 0$. This implies that $w_i = 0$. Hence $T_{\alpha\beta\gamma\delta} = 0$ gives $H;_\alpha = I;_\alpha = J;_\alpha = K;_\alpha = H';_\alpha = I';_\alpha = J';_\alpha = 0$ for $\alpha = 2, 3, 4$. Since the main scalars $H, I, J, K,\ H', I', J'$ are positively homogeneous of degree one in y^i, we have $H;_\alpha = I;_\alpha = J;_\alpha = K;_\alpha = H';_\alpha = I';_\alpha = J';_\alpha = 0$ for $\alpha = 1$. Hence all the eight main scalars $H, I, J, K,\ H', I', J', K'$ are functions of position alone. Therefore we have the following:

Theorem (3.2). If main scalars I and K are equal, and $K' \neq 0$, the $T-$condition for a non-Riemannian Finsler space of four dimensions is equivalent to the fact that the $v-$connection vectors u_i, v_i, and w_i vanishes identically and all the main scalars $H, I, J, K,\ H', I', J', K'$ are functions of position alone.

Remark (3.1). It should be remarked here that the conditions $I \neq K$ and $K';_\alpha = 0$ in theorem (3.1) and $I = K$ and $K' \neq 0$ in theorem (3.2) is only necessary for a Finsler space satisfying $T-$condition to vanish $v-$connection vectors and all the main scalars to be functions of position alone. On the other hand if all the $v-$connection vectors vanish and all the main scalars are functions of position alone, then a four dimensional Finsler space satisfies $T-$condition.

Theorem (3.3)[1]. The tensor T_{hijk} vanishes if and only if the tensor P^i_{jkl} be invariant under any conformal transformation.

In view of theorems (3.2) and (3.3) we have the following:

Theorem (3.4). If $v-$connection vectors u_i, v_i, and w_i of a four dimensional Finsler space M^4 vanishes, and all the main scalars are functions of position alone, then (v) $hv-$curvature tensor P^i_{jkl} of M^4 is conformally invariant under any conformal transformation.

Theorem (3.5)[1]. A Landsberg space remains to be a Landsberg space by any conformal transformation if and only if $T_{hijk} = 0$.

In view of theorems (3.5) and (3.2) we have the following:

Theorem (3.6). If $v-$connection vectors u_i, v_i, and w_i of a four dimensional Finsler space M^4 vanishes, and all the main scalars are functions of position alone, then a Landsberg space remains to be a Landsberg space under any conformal transformation.

4. Landsberg angle in four dimensional Finsler space

In this section we consider Landsberg angle in four dimensional Finsler space M^4. The Landsberg angle θ, ϕ of three dimensional Finsler space with $v-$connection vector $v_i = 0$ is given by [9]

$$\dot{\partial}_i \theta = L^{-1} m_i, \qquad \dot{\partial}_i \phi = L^{-1} n_i. \tag{4.1}$$

The class of four dimensional Finsler spaces with v-connection vectors $u_i = v_i = w_i = 0$ is interested from the view point that we can generalize the Landsberg angle θ, ϕ of three dimensional Finsler space to four dimensions as follows:

We consider the differential equations

$$\dot{\partial}_i \theta = L^{-1} m_i, \qquad \dot{\partial}_i \phi = L^{-1} n_i, \qquad \dot{\partial}_i \psi = L^{-1} p_i, \tag{4.2}$$

Proposition (4.1). If the $v-$connection vectors u_i, v_i and w_i of a four dimensional Finsler space M^4 vanish identically, there exist three scalar fields θ, ϕ and ψ satisfying the differential equation (4.2).

These scalars θ, ϕ, ψ are defined up to additional functions of position only and may be called the Landsberg angles of such a special four dimensional Finsler space.

On account of (2.9) with $u_i = v_i = w_i = 0$ it is easy to show that these equations are completely integrable. The L, θ, ϕ and ψ are regarded as polar coordinates of a kind of the tangent space and

$$\frac{\partial y^i}{\partial L} = l^i, \qquad \frac{\partial y^i}{\partial \theta} = L m^i, \qquad \frac{\partial y^i}{\partial \phi} = L n^i, \qquad \frac{\partial y^i}{\partial \psi} = L p^i, \tag{4.3}$$

are immediately derived.

Let g be the determinant of the fundamental tensor g_{ij} then from $\dot{\partial}_i = 2g\,C_i = 2g\,Cm_i$, it follows that

$$\frac{\partial g}{\partial L} = 0, \qquad \frac{\partial g}{\partial \theta} = 2(LC)g, \qquad \frac{\partial g}{\partial \phi} = 0, \text{ and } \frac{\partial g}{\partial \psi} = 0.$$

Proposition (4.2). The determinant g of the fundamental tensor g_{ij} of a four dimensional non-Riemannian Finsler space with the vanishing $v-$connection vectors u_i, v_i, w_i is of the form $g = te^{2\theta(LC)}$ where t and LC are the functions of position alone. LC is the unified main scalar and θ is the first Landsberg angle.

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