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$\eta-$Ricci Solitons in $\alpha-$Sasakian Manifolds

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Abstract

In this paper we study $\eta-$Ricci solitons in $\alpha-$Sasakian manifolds its shows that a symmetric second order covariant tensors in $\alpha-$Sasakian manifolds is a constant multiple of metric tensor using this it is shown that $L_V g + 2S + 2\mu\eta\otimes\eta$ is parallel, where V is a given vector field then (g, V, μ) is $\eta-$Ricci solitons.

Key Words: Ricci soliton, $\alpha-$Sasakian manifold, Einstein.
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1. Introduction

A Ricci soliton (g, V, λ) is a generalization of an Einstein metric and is defined on a Riemannian manifold (M, g) by

$$(\mathcal{L}_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \tag{1.1}$$

where V is a complete vector field on M, and λ is a constant. The Ricci soliton is said to be shrinking, steady or expanding according as λ is negative, zero and positive respectively.

A η-Ricci soliton [3, 9] is defined on a Riemannian manifold (M, g) by

$$(\mathcal{L}_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta\eta(X)\eta(Y) = 0. \tag{1.2}$$

In [19], Perelman proved that a Ricci soliton on a compact n-manifold is a gradient Ricci soliton. In [23], R. Sharma studied Ricci solitons in K-contact manifolds, where the structure field ξ is Killing and he proved that a complete K-contact gradient soliton is compact Einstein and Sasakian. In [24], M. M. Tripathi studied Ricci solitons in $N(k)$-contact metric and (k, μ) manifolds. In [1], Amadendu Ghosh and Ramesh Sharma studied K-contact metrics as Ricci solitons. In [18], H. G. Nagaraja and C. R. Premalatha studied Ricci

Solitons in f-Kenmotsu Manifolds and 3-dimensional trans-Sasakian manifolds. Recently, C. S. Bagewadi and Gurupadavva Ingalahalli [4] studied Ricci solitons in Lorentzian α-Sasakian Manifolds. Motivated by the above studies on Ricci solitons, in this paper, we study η-Ricci solitons in an α-Sasakian manifolds, where α is some constant.

2. Preliminaries

Let M be an almost contact metric manifold of dimension n equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g, which satisfy

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \tag{2.2}$$

for all $X, Y \in \mathfrak{X}(M)$. An almost contact metric manifold $M(\phi, \xi, \eta, g)$ is said to be α-Sasakian manifold if the following conditions hold:

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X), \tag{2.3}$$

$$\nabla_X \xi = -\alpha\phi X, \quad (\nabla_X \eta)Y = \alpha g(X, \phi Y), \tag{2.4}$$

holds for some non zero constant α on M.

In an α-Sasakian manifold, the following relations hold:

$$R(X, Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y], \tag{2.5}$$

$$R(\xi, X)Y = \alpha^2[g(X, Y)\xi - \eta(Y)X], \tag{2.6}$$

$$\eta(R(X, Y)Z) = \alpha^2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{2.7}$$

$$S(X, \xi) = \alpha^2(n-1)\eta(X), \tag{2.8}$$

$$S(\xi, \xi) = \alpha^2(n-1), \tag{2.9}$$

$$Q\xi = \alpha^2(n-1)\xi, \tag{2.10}$$

for all $X, Y, Z \in \mathfrak{X}(M)$, where R is the Riemannian curvature tensor, S is the Ricci tensor and Q is the Ricci operator.

2.1. **Example.** Let $M = \{(x, y, z) \in R^3\}$. Let (E_1, E_2, E_3) be linearly independent vector fields given by

$$E_1 = e^x \frac{\partial}{\partial y}, \quad E_2 = e^x \left[\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}\right], \quad E_3 = \frac{\partial}{\partial z}. \tag{2.11}$$

Let g be the Riemannian metric defined by $g(E_1, E_2) = g(E_2, E_3) = g(E_1, E_3) = 0, g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1$, where g is given by

$$g = \frac{1}{e^{2x}}[(1 - 4e^{2x}y^2)dx \otimes dx + dy \otimes dy + e^{2x}dz \otimes dz].$$

Let η be the 1-form defined by $\eta(U) = g(U, E_3)$ for any $U \in \mathfrak{X}(M)$. Let ϕ be the $(1, 1)$ tensor field defined by $\phi E_1 = E_2$, $\phi E_2 = -E_1$, $\phi E_3 = 0$. Then using the linearity of ϕ and g yields that $\eta(E_3) = 1$, $\phi^2 U = -U + \eta(U)E_3$ and $g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$ for any vector fields $U, W \in \mathfrak{X}(M)$. Thus for $E_3 = \xi$, (ϕ, ξ, η, g) defines a Sasakian structure on M. By definition of Lie bracket, we have

$$[E_1, E_2] = -e^x E_1 + 2e^{2x} E_3, \quad [E_1, E_3] = [E_2, E_3] = 0.$$

Let ∇ be the Levi-Civita connection with respect to above metric g Koszula formula is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \tag{2.12}$$

Then

$$\begin{aligned} \nabla_{E_1} E_1 &= e^x E_2, & \nabla_{E_2} E_2 &= 0, & \nabla_{E_3} E_3 &= 0, \\ \nabla_{E_1} E_2 &= -\, e^x E_1 + e^{2x} E_3, & \nabla_{E_2} E_1 &= -e^{2x} E_3, & \nabla_{E_2} E_3 &= e^{2x} E_1, \\ \nabla_{E_1} E_3 &= -\, e^{2x} E_2, & \nabla_{E_3} E_1 &= -\, e^{2x} E_2, & \nabla_{E_3} E_2 &= e^{2x} E_1. \end{aligned} \tag{2.13}$$

Clearly (ϕ, ξ, η, g) structure is an $\alpha-$Sasakian structure and satisfy,

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X), \qquad \nabla_X \xi = -\alpha\phi X, \tag{2.14}$$

where $\alpha = e^{2x} \neq 0$. Hence (ϕ, ξ, η, g) structure defines α-Sasakian structure. Thus M equipped with α-Sasakian structure is a α-Sasakian manifold. The tangent vectors X and Y to M are expressed as linear combination of E_1, E_2, E_3, that is $X = \sum_{i=1}^{3} a_i E_i$ and $Y = \sum_{i=1}^{3} b_i E_i$, where a_i and $b_i (i = 1, 2, 3)$ are scalars.

On α-Sasakian manifold (M, g), we have

$$(\mathcal{L}_V g)(X, Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V) \tag{2.15}$$

where ∇ denotes the Levi-Civita connection of M. Hence if (M, g) is a η-Ricci soliton with potential vector field V, then (1.2) and (2.15), we have

$$2S(X, Y) = -g(\nabla_X V, Y) - g(X, \nabla_Y V) - 2\lambda g(X, Y) - 2\mu\eta\eta(X)\eta(Y). \tag{2.16}$$

By taking $X = Y = e_i$ where e_i is an orthonormal basis and $1 \leq i \leq n$, then we have

$$\int_M [divV + r + n\lambda + \mu] = 0. \tag{2.17}$$

On integrating the above equation we have by Green's theorem $\int divV = 0$ and for scalar curvature r, then we have

$$(r + n\lambda + \mu)Vol(M) = 0. \tag{2.18}$$

The above equation implies that

$$r = -(n\lambda + \mu). \tag{2.19}$$

For Ricci solitons $\mu = 0$, then

$$\lambda = -\frac{r}{n}. \tag{2.20}$$

In α-Sasakian manifolds scalar curvature $r = \alpha^2(n-1)$, we have

$$\lambda = -\frac{\alpha^2(n-1)}{n} < 0. \tag{2.21}$$

Hence, we state the following:

Theorem 2.1. A $\eta-$Ricci soliton in an α-Sasakian is shrinking.

Corollary 2.1. If a metric g in an α-Sasakian manifold is a η-Ricci soliton with $V = \xi$ then it is η-Einstein.

Proof. Putting $V = \xi$ in (1.2), then we have

$$(\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0 \tag{2.22}$$

where

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0 \tag{2.23}$$

Substituting (2.23) in (2.22), then we get the result.

Proposition 2.1. If an α-Sasakian manifold is a η-Ricci soliton with V pointwise collinear with ξ, then V is a constant multiple of ξ and the manifold is Einstein.

Proof.

$$(\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta\eta(X)\eta(Y) = 0 \tag{2.24}$$

where

$$(\mathcal{L}_V g)(X, Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V). \tag{2.25}$$

Substituting (2.25) in (2.24), then we have

$$g(\nabla_X V, Y) + g(X, \nabla_Y V) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta\eta(X)\eta(Y) = 0. \tag{2.26}$$

Putting $V = a\xi$ in (2.26), we have

$$(Xa)\eta(Y) + (Ya)\eta(X) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta\eta(X)\eta(Y) = 0. \tag{2.27}$$

Putting $X = Y = \xi$ in (2.27), we have

$$(\xi a) + \alpha^2(n-1) + \lambda + \mu = 0. \tag{2.28}$$

Again putting $X = \xi$ in (2.27), we have

$$(Ya) = [-\alpha^2(n-1) - \lambda - \mu]\eta(Y). \tag{2.29}$$

Equation (2.29) implies that

$$da = [-\alpha^2(n-1) - \lambda - \mu]\eta. \tag{2.30}$$

Applying d on both sides

$$d^2 a = [-\alpha^2(n-1) - \lambda - \mu]d\eta. \tag{2.31}$$

Since $d^2a = 0$ but $d\eta$ is nowhere vanishing. Therefore, $-\lambda - \alpha^2(n-1) - \mu = 0$ which implies $da = 0$ that is, a is constant. On the above hence we state that

Theorem 2.2. On an α-Sasakian manifold, the contact form η is closed if and only if ξ is integrable and the Nijenhuis tensor field of the structural endomorphism ϕ vanishes identically.

Proof. From (2.4), we have

$$\begin{aligned}(d\eta)(X,Y) &= \frac{1}{2}[X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y])] \\ &= \frac{1}{2}[g(Y,\nabla_X\xi) - g(X,\nabla_Y\xi)] = -\alpha g(\phi X, Y).\end{aligned} \tag{2.32}$$

If ξ is integrable then $d\eta = 0$.
Nijenhuis tensor field of the endomorphism is given by

$$\begin{aligned}N_\phi(X,Y) &= \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] \\ &= \phi^2\{\nabla_X Y - \nabla_Y X\} + \{\nabla_{\phi X}\phi Y - \nabla_{\phi Y}\phi X\} \\ &- \phi\{\nabla_{\phi X} Y - \nabla_Y \phi X\} - \phi\{\nabla_X \phi Y - \nabla_{\phi Y} X\} = 0.\end{aligned} \tag{2.33}$$

3. Parallel symmetric second order tensors and Ricci Solitons in $\alpha-$Sasakian manifolds

Fix h a symmetric tensor field of $(0,2)$-type which we suppose to be parallel with respect to ∇ that is $\nabla h = 0$. Applying the Ricci identity [20]

$$\nabla^2 h(X,Y;Z,W) - \nabla^2 h(X,Y;W,Z) = 0, \tag{3.1}$$

we obtain the relation

$$h(R(X,Y)Z,W) + h(Z,R(X,Y)W) = 0. \tag{3.2}$$

Replacing $Z = W = \xi$ in (3.2) and by using (2.5) and by the symmetry of h, we have

$$2\alpha^2[\eta(Y)h(X,\xi) - \eta(X)h(Y,\xi)] = 0. \tag{3.3}$$

Put $X = \xi$ in (3.3) and by virtue of (2.1), we have

$$2\alpha^2[\eta(Y)h(\xi,\xi) - h(Y,\xi)] = 0. \tag{3.4}$$

Since $\alpha^2 \neq 0$, it results

$$h(Y,\xi) = \eta(Y)h(\xi,\xi). \tag{3.5}$$

Let us call a regular α-Sasakian manifolds with $\alpha^2 \neq 0$ and remark that α-Sasakian manifold is regular, where regularity means the nonvanishing of the Ricci curvature with respect to the generator of α-Sasakian manifolds.

Definition 3.1. ξ is called semi-torse forming vector field for α-Sasakian manifold if, for all vector fields X :

$$R(X,\xi)\xi = 0. \tag{3.6}$$

From (2.5), we have $R(X,\xi)\xi = \alpha^2[X - \eta(X)\xi]$ and therefore, if $X \in ker\xi = \xi^{\perp}$, then $R(X,\xi)\xi = \alpha^2 X$ and we obtain:

Proposition 3.1. For an α-Sasakian manifold the following are equivalent:

(1) is regular,
(2) ξ is not semi-torse forming,
(3) $S(\xi,\xi) \neq 0$, that is, ξ is non-degenerate with respect to S,

Now, differentiating the equation (3.5) covariantly with respect to X, we have

$$\begin{aligned}(\nabla_X h)(Y,\xi) + h(\nabla_X Y,\xi) + h(Y,\nabla_X\xi) &= [(\nabla_X\eta)(Y) + \eta(\nabla_X Y)]h(\xi,\xi) \\ &+ \eta(Y)[(\nabla_X h)(Y,\xi) + 2h(\nabla_X\xi,\xi)].\end{aligned} \tag{3.7}$$

By using the parallel condition $\nabla h = 0$, $\eta(\nabla_X\xi) = 0$ and (3.5) in (3.7), we have

$$h(Y,\nabla_X\xi) = (\nabla_X\eta)(Y)h(\xi,\xi). \tag{3.8}$$

By using (2.4) in (3.8), we get

$$-\alpha h(Y,\phi X) = \alpha g(X,\phi Y)h(\xi,\xi). \tag{3.9}$$

Replacing $X = \phi X$ in (3.9), we get

$$\alpha[h(Y,X) - g(Y,X)h(\xi,\xi)] = 0. \tag{3.10}$$

Clearly α is a nonzero smooth function in α-Sasakian manifold this implies that

$$h(X,Y) = g(X,Y)h(\xi,\xi), \tag{3.11}$$

the above equation implies that $h(\xi,\xi)$ is a constant, via (3.5). Now by considering the above condition we state the following theorem:

Theorem 3.1. A symmetric parallel second order covariant tensor in an α-Sasakian manifold is a constant multiple of the metric tensor.

Theorem 3.2. Let M be a α-Sasakian manifold, the symmetric (0, 2)-tensor field $h := (\mathcal{L}_\xi g)(X,Y) + 2S(X,Y) + 2\mu\eta(X)\eta(Y)$ is parallel with respect to the Levi-Civita connection associated to g. Then (g,ξ,λ,μ) yields an η-Ricci soliton.

Proof. Assume $h(\xi,\xi) = (\mathcal{L}_\xi g)(\xi,\xi) + 2S(\xi,\xi) + 2\mu\eta(\xi)\eta(\xi)$. Now (2.22), can be written in form

$$h(X,Y) = -2\lambda g(X,Y). \tag{3.12}$$

that is,

$$\lambda = \frac{-1}{2}h(\xi,\xi). \tag{3.13}$$

Therefore, $(\mathcal{L}_\xi g)(\xi,\xi) + 2S(\xi,\xi) + 2\mu\eta(\xi)\eta(\xi) = -2\lambda g(\xi,\xi)$.

If $\mu = 0$, then $(\mathcal{L}_\xi g)(\xi,\xi) + 2S(\xi,\xi) = -2\lambda g(\xi,\xi)$. Hence we conclude that

Corollary 3.1. On a α-Sasakian manifold the symmetric (0, 2)-tensor field $h := (\mathcal{L}_\xi g)(\xi,\xi) + 2S(\xi,\xi)$ is parallel with respect to the Levi-Civita connection associated to g, then the η-Ricci soliton relation defines a Ricci soliton on M.

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