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Ricci Flow Equations on Special Finsler Space

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Abstract

In this paper, we deal with one of the special Finsler spaces such as C_2-like space and find out Ricci flow equations on C_2-like space with $(\alpha, \beta)-$metric.

Key Words: Ricci flow, (α, β)-metric, C_2-like space.
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1. Introduction

Ricci flow is a means by which one can take an arbitrary Riemannian manifold and smooth out the geometry of that manifold to make it look more symmetric. It has proven to be a very useful tool in understanding the topology of such manifolds.

The Ricci flow theory became a very powerful method in understanding the geometry and topology of Riemannian manifolds ([3], [7]-[9]). The most important achievement of this theory was the geometrization conjecture of Thurston. One consequence of this conjecture is the Poincare conjecture. This conjecture was formulated by Henri Poincare [4] and proved by Perelman ([7]-[9]). The proof of Poincare conjecture based on a detailed analysis of Ricci flow surgery is one of the most impressive recent achievement of modern mathematics.

S. Vacaru ([12]-[18]) studied on nonholonomic Ricci flows, evolution equations and dynamics, exact solutions in gravity, symmetric and non symmetric metrics, the entropy of Lagrange-Finsler spaces and Ricci flows, spectral functionals, nonholonomic Dirac operators and non commutative Ricci flows, Fractional nonholonomic Ricci flows, Nonholonomic ricci flows and parametric deformations of the solitonic pp-waves and schwarzschild solutions. A. Thayebi, E. Peyghan and B. Najafi [11] studied on Ricci flow equation on $(\alpha, \beta)-$metrics.

R. S. Hamilton [3] introduced the following geometric evolution equation for a Riemannian metric g_{ij} and the corresponding Ricci curvature tensor Ric_{ij}

$$\frac{d}{dt}(g_{ij}) = -2Ric_{ij},\ g(t=0) = g_0 \tag{1.1}$$

is known as the un-normalized Ricci flow in Riemannian geometry. Hamilton showed that there is a unique solution to this equation for an arbitrary smooth metric on a closed manifold over a sufficiently short time.

In this paper, we deal with one of the special Finsler spaces such as C_2-like space and find out un-normal Ricci flow and normal Ricci flow equations on C_2-like space with $(\alpha, \beta)-$metric.

2. Preliminaries

Definition 2.1. A Finsler metric is a scalar field $L(x, y)$ which satisfies the following three conditions:

(i) It is defined and differential for any point of $TM^n \setminus \{0\}$,
(ii) It is positively homogeneous of first degree in y^i, that is,
$L(x, \lambda y) = \lambda L(x, y)$, for any positive number λ,
(iii) It is regular, that is,
$g_{ij}(x, y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$,
constitute the regular matrix g_{ij}, where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$.

The manifold M^n equipped with a fundamental function $L(x, y)$ is called Finsler space $F^n = (M^n, L)$.

The concept of the $(\alpha, \beta)-$metric was introduced in 1972 by M. Matsumoto and has been studied by M. Hashiguchi, Y. Ichijyo, S. Kikuchi, C. Shibata and others ([5], [6] and [10]).

Definition 2.2. The Finsler space $F^n = (M^n, L)$ is said to have an $(\alpha, \beta)-$metric if L is a positively homogeneous function of degree one in two variables $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_i(x)y^i$, where α is a Riemannian metric and β is differential $1-$form.

A deformation of Finsler metrics means a $1-$parameter family of metrics $g_{ij}(x, y, t)$, such that $t \in [-\epsilon, \epsilon]$ and $\epsilon > 0$ is sufficiently small. For such a metric $w = u_i dx^i$, the volume element as well as the connections attached to it depend on t. The same equation can be used in the Finsler setting. Another Ricci flow equation can also be used instead of this tensor evolution equation [2]. By contracting $\frac{d}{dt}g_{ij} = -2Ric_{ij}$ with y^i and y^j gives, via Euler's theorem, we get

$$\frac{\partial L^2}{\partial t} = -2L^2 R,$$

where $R = \frac{1}{L^2}\, Ric$. That is,

$$d\ \log\, L = -R,\ \ L(t=0) = L_0.$$

This scalar equation directly addresses the evolution of the Finsler metric L and makes geometrical sense on both the manifold of nonzero tangent vectors TM_0 and the manifold of rays. It is therefore suitable as an un-normalized Ricci flow for Finsler geometry.

By using the elegance work of Akbar-Zadeh in [1], Bao proposed the following normalised Ricci flow equation for Finsler metrics

$$\frac{d}{dt}\log\, L = -R + \frac{1}{Vol(SM)}\int_{SM} R\ dV,\ \ L(t=0) = L_0,$$

where the underlying manifold M is compact [2].

It is noted that [11], Chern had asked whether every smooth manifold admits a Ricci-constant Finsler metric? The weaker case of this question is that whether every smooth manifold admits a Einstein Finsler metric? His question has already been settled in the affirmative for dimension 2 because, by a construction of Thurstons, every Riemannian metric on a two-dimensional manifold admits a complete Riemannian metric of constant Gaussian curvature.

Let M be an n-dimensional C^∞ manifold, T_xM be the tangent space at $x \in M$ and $TM = \cup_{x\in M} T_xM$ be the tangent bundle of M. Let $x \in M$ and $L_x = L|_{T_xM}$. To measure the non-Euclidean feature of L_x, define $C_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$C_y(u,v,w) = \frac{1}{2}\frac{d}{dt}\left[g_{y+tw}(u,v)\right]|_{t=0}, \quad u,v,w \in T_xM.$$

The family $C = \{C_y\}_{y\in TM_0}$ is called the Cartan torsion. It is known that $C = 0$ if and only if L is Riemannian.

For $y \in T_xM_0$, define mean Cartan torsion I_y by $I_y(u) = I_i(y)u^i$, where $I_i = g^{jk}C_{ijk}$, $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ and $u = u^i \frac{\partial}{\partial x^i}|_x$. By Deicke's theorem, L is Riemannian if and only if $I_y = 0$.

A Finsler metric L is called C_2-like if its Cartan tensor is given by

$$C_{ijk} = \frac{1}{\|I\|^2} I_i I_j I_k.$$

3. Un-normal Ricci flow equation on C_2-like space with (α, β)-metrics

Here, we study (α, β)-metrics satisfying un-normal Ricci flow equation. First, we prove the following lemmas.

Lemma 3.1. Let L_t be a deformation of an (α, β)-metric L, which is C_2-like, on a manifold M of dimension $n \geq 3$. Then the variation of Cartan tensor is given by the following

$$C'_{ijk} I^i I^j I^k = -2R\|I\|^4 - \frac{1}{2} L^2 R_{,i,j,k} I^i I^j I^k - 3\|I\|^2 I^m R_{,m}, \tag{3.1}$$

where $\|I\|^2 = I_m I^m$.

Proof. First, assume that L_t be a deformation of a Finsler metric on a two-dimensional manifold M satisfies Ricci flow equation, that is,

$$\frac{d}{dt} g_{ij} = g'_{ij} = -2Ric_{ij}, \quad d\ logL = \frac{L'}{L} = -R, \tag{3.2}$$

where $R = \frac{1}{L^2} Ric$. By definition of Ricci tensor, we have

$$\begin{aligned} Ric_{ij} &= \frac{1}{2}[RL^2]_{y^i y^j} \\ &= Rg_{ij} + \frac{1}{2} L^2 R_{,i,j} + R_{,i} y_j + R_{,i} y_i, \end{aligned} \tag{3.3}$$

where $R_{,i} = \frac{\partial R}{\partial y^i}$ and $R_{,i,j} = \frac{\partial^2 R}{\partial y^i \partial y^j}$. Taking a vertical derivative of (3.3) and using $y_{i,j} = g_{ij}$ and $LL_k = y_k$ yields

$$\begin{aligned} Ric_{ij,k} &= 2RC_{ijk} + \frac{1}{2} L^2 R_{,i,j,k} + \{g_{jk} R_{,i} + g_{ij} R_{,k} + g_{ki} R_{,j}\} \\ &\quad + \{R_{,j,k} y_i + R_{,i,j} y_k + R_{,k,i} y_j\}. \end{aligned} \tag{3.4}$$

Contracting (3.4) with $I^i I^j I^k$ and using $y_i I^i = y^i I_i = 0$ implies that

$$Ric_{ij,k} I^i I^j I^k = 2RC_{ijk} I^i I^j I^k + \frac{1}{2} L^2 R_{,i,j,k} I^i I^j I^k + 3\|I\|^2 I^m R_{,m}. \tag{3.5}$$

The Cartan tensor of an (α, β)-metric on n-dimensional manifold M is given by

$$C_{ijk} = \frac{1}{\|I\|^2} I_i I_j I_k. \tag{3.6}$$

Multiplying (3.6) with $I^i I^j I^k$ yields

$$C_{ijk} I^i I^j I^k = \|I\|^4. \tag{3.7}$$

Then by (3.5) and (3.7), we get

$$\begin{aligned} Ric_{ij,k}I^iI^jI^k &= 2R\|I\|^4 + \frac{1}{2}L^2R_{,i,j,k}I^iI^jI^k \\ &\quad +3\|I\|^2I^mR_{,m}. \end{aligned} \tag{3.8}$$

On the other hand, since L_t satisfies Ricci flow equation, then

$$\begin{aligned} C'_{ijk} &= \frac{1}{2}\frac{\partial g'_{ij}}{\partial y^k} \\ &= \frac{1}{2}\frac{\partial(-2Ric_{ij})}{\partial y^k} \\ &= -Ric_{ij,k}. \end{aligned} \tag{3.9}$$

By (3.8) and (3.9), we get (3.1).

Lemma 3.2. Let L_t be a deformation of an (α,β)-metric L, which is C_2-like, on a manifold M of dimension $n \geq 3$. Then $C'_{ijk}I^iI^jI^k$ is a factor of $\|I\|^2$.

Proof. Since $g^{ij}g_{jk} = \delta^i_k$, we have

$$\begin{aligned} &\left(g^{ij}g_{jk}\right)' = 0 \\ &\Rightarrow g'^{ij}g_{jk} + g^{ij}g'_{jk} = 0 \\ &\Rightarrow g'^{ij}g_{jk} + g^{ij}(-2Ric_{jk}) = 0 \\ &\Rightarrow g'^{ij}g_{jk} - 2g^{ij}Ric_{jk} = 0, \end{aligned} \tag{3.10}$$

or equivalently, $\left(g^{ij}\right)' g_{jk} = 2g^{ij}Ric_{jk}$.
Contracting with g^{lk} gives

$$\left(g^{il}\right)' = 2Ric^{il}. \tag{3.11}$$

Then, we have

$$\begin{aligned} I'_i &= \left(g^{jk}c_{ijk}\right)' \\ &= \left(g^{jk}\right)' c_{ijk} + g^{jk}\left(c_{ijk}\right)' \\ &= 2Ric^{jk}c_{ijk} + g^{jk}\left(-Ric_{ij,k}\right) \\ &= Ric^{jk}\frac{\partial g_{ij}}{\partial y^k} - \left(g^{jk}Ric_{jk}\right)_{,i} + g^{jk}{}_{,i}Ric_{jk}. \end{aligned} \tag{3.12}$$

Since

$$-g^{jk}Ric_{ij,k} = -\left(g^{jk}Ric_{jk}\right)_{,i} + g^{jk}{}_{,i}Ric_{jk},$$

we have

$$\begin{aligned} I'_i &= Ric^{jk} g_{jk,i} - \left(g^{jk} Ric_{jk}\right)_{,i} + g^{jk}{}_{,i} Ric_{jk} \\ &= -\left(g^{jk} Ric_{jk}\right)_{,i} \\ &= -\rho_i, \end{aligned} \tag{3.13}$$

where $\rho = g^{jk} Ric_{jk}$ and $\rho_i = \frac{\partial \rho}{\partial y^i}$. Thus

$$\begin{aligned} I'^i &= \left(g^{ij} I_j\right)' \\ &= \left(g^{ij}\right)' I_j + g^{ij} I'_j \\ &= 2Ric^{ij} I_j + g^{ij}(-\rho_j) \\ &= 2Ric^{ij} I_j - \rho^i. \end{aligned} \tag{3.14}$$

The variation of $y_i = LL_{y^i}$ with respect to t is given by

$$y'_i = -2Ric_{im} y^m.$$

Therefore, we can compute the variation of angular metric h_{ij} as follows

$$\begin{aligned} h'_{ij} &= \left(g_{ij} - L^{-2} y_i y_j\right)' \\ &= (g_{ij})' - \left(L^{-2} y_i y_j\right)' \\ &= -2Ric_{ij} - \left\{L^{-2}\left[y'_i y_j + y_i y'_j\right] + y_i y_j \left(L^{-2}\right)'\right\} \\ &= -2Ric_{ij} - L^{-2}\left[-2Ric_{im} y^m y_j - 2Ric_{jm} y^m y_i\right] - 2L^{-2} R y_i y_j \\ &= -2Ric_{ij} + 2(h_{ij} - g_{ij})R + 2\left(Ric_{im} L^{-1} y_j\right) L^{-1} y^m \\ &\quad + 2\left(Ric_{jm} L^{-1} y_i\right) L^{-1} y^m \\ &= -2Ric_{ij} + 2\left(h_{ij} - g_{ij}\right) R + 2\left(Ric_{im} l_j + Ric_{jm} l_i\right) l^m, \end{aligned} \tag{3.15}$$

where $l_i = L^{-1} y_i$ and $l^m = L^{-1} y^m$. Thus, we consider the variation of Cartan tensor

$$\begin{aligned} C'_{ijk} &= \left[\frac{1}{\|I\|^2} I_i I_j I_k\right]' \\ &= \frac{\|I\|^2 \left(I_i I_j I_k\right)' - I_i I_j I_k \left(\|I\|^2\right)'}{\|I\|^4} \\ &= \frac{\left(I'_i I_j I_k + I'_j I_i I_k + I'_k I_i I_j\right)}{\|I\|^2} - \frac{C_{ijk}\left(I'^m I_m + I^m I'_m\right)}{\|I\|^2} \\ &= -\frac{\left(\rho_i I_j I_k + \rho_j I_i I_k + \rho_k I_i I_j\right)}{\|I\|^2} - \frac{\left(I'^m I_m + I^m I'_m\right) C_{ijk}}{\|I\|^2}. \end{aligned} \tag{3.16}$$

Multiplying (3.16) with $I^i I^j I^k$ gives

$$\begin{aligned} C'_{ijk} I^i I^j I^k &= -\frac{\left(\rho_i I^i + \rho_j I^j + \rho_k I^k\right) \|I\|^4}{\|I\|^2} \\ &\quad -\frac{(I'^m I_m + I^m I'_m)}{\|I\|^2} \times \frac{1}{\|I\|^2} \times I_i I_j I_k \times I^i I^j I^k \\ &= -\frac{3\|I\|^4 \rho_m I^m}{\|I\|^2} - \frac{\left[\left(2Ric^{mj} I_j - \rho^m\right) I_m + I^m \left(-\rho_m\right)\right]}{\|I\|^2} \times \frac{\|I\|^6}{\|I\|^2} \\ &= \|I\|^2 \left\{\rho^m I_m - 2\left(Ric^{mj} I_m I_j + \rho_m I^m\right)\right\}, \end{aligned} \tag{3.17}$$

which implies $C'_{ijk} I^i I^j I^k$ is a factor of $\|I\|^2$. This completes the proof.

Next, we prove the following main theorem.

Theorem 3.1. Suppose that L is an (α, β)-metric on M, which is C_2-like, then every deformation L_t of the metric L satisfying un-normal Riccci flow equation is an Einstein metric.

Proof. By virtue of lemma and lemma , $R_{,i,j,k} I^i I^j I^k$ is a factor of $\|I\|^2$. Since $R_{,i,j,k} I^i I^j I^k$ is a factor of $\|I\|^2$, multiplying it with y^k or y^j implies $R_{,i} = 0$. It means that $R = R(x)$ and then L_t is an Einstein metric.

4. Normal Ricci flow equation on C_2-like space with (α, β)-metrics

If M is a compact manifold, then $S(M)$ is compact and we can normalize the Ricci flow equation by requiring that the flow keeps the volume of SM constant. Recalling the Hilbert form $w = L_{y^i} dx^i$, that volume is

$$Vol_{SM} = \int_{SM} \frac{(-1)^{\frac{n(n-1)}{2}}}{(n-1)!} w \wedge (dw)^{n-1} = \int_{SM} dV_{SM}.$$

During the evolution, L, w and consequently the volume form dV_{SM} and the volume Vol_{SM}, all depend on t. On the other hand, the domain of integration SM, being the quotient space of TM_0 under the equivalence relation $z \sim y$, $z = \lambda y$ for some $\lambda > 0$, is totally independent of any Finsler metric and hence does not depend on t. We have

$$\frac{d}{dt}(dV_{SM}) = \left[g_{ij} \frac{d}{dt} g_{ij} - n \frac{d}{dt} logL\right] dV_{SM}.$$

A normalized Ricci flow for Finsler metrics is proposed by Bao as follows

$$\frac{d}{dt} logL = -R + \frac{1}{Vol(SM)} \int_{SM} R\, dV, \quad L(t=0) = L_0, \tag{4.1}$$

where the underlying manifold M is compact. Now, we let $Vol(SM) = 1$. Then all of Ricci constant metrics are exactly the fixed points of the above flow. Let

$$Ric_{ij} = \frac{1}{2}\left(L^2R\right)_{.y^i.y^j}$$

and differentiating (4.1) with respect to y^i and y^j, the following normal Ricci flow tensor evaluation equation is concluded.

$$\frac{d}{dt}g_{ij} = -2Ric_{ij} + \frac{2}{Vol(SM)}\int_{SM} R\ dV g_{ij}, \quad g(t=0) = g_0. \tag{4.2}$$

Starting with any familiar metric on M as the initial data L_0, we may deform it using the proposed normalized Ricci flow, in the hope of arriving at a Ricci constant metric.

Theorem 4.1. Suppose that L is an (α, β)-metric on M, which is C_2-like, then every deformation L_t of the metric L satisfying normal Riccci flow equation is an Einstein metric.

Proof. Consider Finsler surfaces which satisfy the normal Ricci flow equation. Then

$$\begin{aligned} \frac{dg_{ij}}{dt} &= -2Ric_{ij} + 2\int_{SM} R\ dV g_{ij}, \\ d\ logL &= \frac{L'}{L} = -R + \int_{SM} R\ dV. \end{aligned} \tag{4.3}$$

By the same argument in the un-normal Ricci flow case, we can calculate the variation of mean Cartan tensor as follows

$$\begin{aligned} I'_i &= \left(g^{jk}C_{ijk}\right)' \\ &= \left(g^{jk}\right)' C_{ijk} + g^{jk}\left(C_{ijk}\right)' \\ &= \left[2Ric^{jk} - 2\int_{SM} R\ dV g_{jk}\right] C_{ijk} + g^{jk}\left[Ric_{jk,i} + 2\int_{SM} R\ dV C_{ijk}\right] \\ &= -\rho_i. \end{aligned} \tag{4.4}$$

Then we have

$$\begin{aligned} I'^i &= \left(g^{ij}I_j\right)' + g^{ij}I'_j \\ &= \left[2Ric^{ij} - 2\int_{SM} R\ dV g_{ij}\right] I_j - g^{ij}\rho_j. \end{aligned} \tag{4.5}$$

As the similar way that we used in un-normal Ricci flow, it follows that

$$\begin{aligned} C'_{ijk} &= \left[\frac{1}{\|I\|^2} I_i I_j I_k\right]' \\ &= -\frac{(I'^m I_m + I^m I'_m)\, C_{ijk}}{\|I\|^2} - \frac{(\rho_i I_j I_k + \rho_j I_i I_k + \rho_k I_i I_j)}{\|I\|^2}. \end{aligned} \tag{4.6}$$

Contracting it with $I^i I^j I^k$, we can say $C'_{ijk} I^i I^j I^k$ is a factor of $\|I\|^2$. By lemma , we deduce that $R_{,i,j,k} I^i I^j I^k$ is a factor of $\|I\|^2$. By the same argument, it results that every deformation L_t of the metric L satisfying normal Ricci flow equation is an Einstein metric.

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