J. T. S.
Special Vol. 8 (2014), pp.131-137
https://doi.org/10.56424/jts.v8i01.10551

Projectively Flat Finsler Space With Special $(\alpha, \beta)-$Metric

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(Received: November 11, 2013)

Abstract

In this paper, we consider special $(\alpha, \beta)-$metric $L = \alpha + \frac{2\beta^2}{\alpha} - \frac{1}{3}\frac{\beta^4}{\alpha^3} + \epsilon\beta$ $(\epsilon \neq 0)$ constant, and discuss the condition for a Finsler space with special $(\alpha, \beta)-$metric to be projectively flat on the basis of Matsumoto's results where α is a Riemannian metric, β is a differential 1-form.

Key Words: Finsler Space, $(\alpha, \beta)-$metrics, Projectively flat, Matsumoto metric, Riemannian metric.
2010 AMS Subject Classification: 53C60, 53B40.

1. Introduction

Let $F^n = (M^n, L)$ be an $n-$dimensional Finsler space, that is, an n-dimensional differential manifold M^n equipped with a fundamental function $L(x, y)$. The concept of an $(\alpha, \beta)-$metric, $L(\alpha, \beta)$ was introduced by M. Matsumoto in 1972 [10] and has been studied by many Finsler geometers. Physicists are also interested in these metrics. They seek for some non-Riemannian models for space time. For example, by using $(\alpha, \beta)-$metrics, G. S. Asanov introduced Finsleroid-Finsler spaces and formulated pseudo-Finsleroid gravitational field. Indetailed investigation was done by M. Hashiguchi and Y. Ichijyo [8], have worked on some special (α, β)-metric. II-Yong Lee and Hong-Suh Park have studied Finsler spaces with infinite series $(\alpha, \beta)-$metric.

A change $L \to \bar{L}$ of a Finsler metric on a same underlying manifold is called projective, if any geodesic on (M^n, L) remains to be a geodesic in $(M^n, \bar{L})$ and vice versa. A Finsler space is called *projectively flat*, or with *rectilinear geodesic*, if the space is coveres by coordinate neighborhoods in which the geodesic can be represented by $(n-1)$ linear equations of the coordinates. Such a coordinate system is called *rectilinear*. The condition for a Finsler space to be projectively flat was studied by L. Berwald [7]. The condition for a Finsler space with (α,β)-metric to be projectively flat was studied by M. Matsumoto [9]. B. D. Kim was studied on the projectively flat Finsler space with (α,β)-metric [3]. Then H. S. Park, H. Y. Park, I. Y. Lee were investigated the projective flatness of Finsler space with some particular (α,β)-metrics [4][5][6]. Later on many authors worked on projective flatness of (α,β)-metric [12] [13][14]. Recently Narasimhamurthy S. K. Latha kumara G. N. and C. S. Bagewadi were studied the projectively flat Finsler space with some special (α,β)- metric [11].

The purpose of the present paper is to consider the projective flatness of Finsler space with an special (α,β)-metric.

2. Preliminaries

Definition 2.1. A Finsler metric is a scalar field $L(x,y)$ which satisfies the following three conditions:

(1) It is defined and differential for any point of $TM^n\backslash\{0\}$,
(2) It is positively homogeneous of first degree in y^i, that is, $L(x,\lambda y)=\lambda L(x,y)$, for any positive number λ,
(3) It is regular, that is, $g_{ij}(x,y)=\frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$, constitute the regular matrix g_{ij}, where $\dot{\partial}_i=\frac{\partial}{\partial y^i}$.

The manifold M^n equipped with a fundamental function $L(x,y)$ is called Finsler space $F^n=(M^n,L)$.

There is a class of Finsler metrics defined by a Riemannian metric and 1-form on a manifold, which is relatively simple with interesting curvature properties called $(\alpha,\beta)-$metrics and these metrics are computable.

Definition 2.2. The Finsler space $F^n=(M^n,L)$ is said to be have an $(\alpha,\beta)-$metric if L is a positively homogeneous function of degree one in two variables $\alpha=\sqrt{a_{ij}y^iy^j}$ and $\beta=b_i(x)y^i$, where α is a Riemannian metric and β is differentiable 1-form.

Example: Consider,

$$\alpha = \frac{(1-\epsilon^2)\langle x,y\rangle^2+\epsilon|y|^2(1+\epsilon|x|^2)}{(1+\epsilon|x|^2)},$$

$$\beta = \frac{\sqrt{(1-\epsilon^2)}(x,y)}{(1+\epsilon|x|^2),}$$

where $x\epsilon R^2$, $y\epsilon R^2$, $\epsilon\ \epsilon(0,1)$, then

$$\|\beta\| = \sqrt{(1-\epsilon^2)}\sqrt{\frac{|x|^2}{+|x|^2}} < 1.$$

Hence $F = \alpha + \beta$ is a Randers metric.

An $(\alpha,\beta)-$metric is expressed in the following form

$$L = \alpha\phi(s), \quad s = \frac{\beta}{\alpha},$$

where $\phi = \phi(s)$ is a C^∞ positive function on an open interval (b_0, b_0). The norm $\|\beta_x\|_\alpha$ of β with respect to α is defined by,

$$\|\beta_x\|_\alpha = sup_{y\epsilon T_xM}\beta(x,y), \alpha(x,y) = \sqrt{a_ij(x)b_i(x)b_j(x)}.$$

In order to define L, β must satisfy the condition $\|\beta_x\|_\alpha < b_0$ for all $x\epsilon M$.

Definition 2.3. A Finsler metric is called a projectively flat metric if it is projectively related to a locally Minkowskian metric.

The study of some well known $(\alpha,\beta)-$metrics are Randers metric $\alpha+\beta$, Kropina metric $\frac{\alpha^2}{\beta}$ and generalized Kropina metric $\frac{\alpha^{(m+1)}}{\beta^m}$ etc., have greatly contributed to the growth of Finsler geometry and its applications to theory of relativity.

The derivative of the $(\alpha,\beta)-$metric with respect to α and β are given by,

$$\begin{aligned} L_\alpha &= \frac{\partial L}{\partial \alpha}, \\ L_\beta &= \frac{\partial L}{\partial \beta}, \\ L_{\alpha\alpha} &= \frac{\partial L_\alpha}{\partial \alpha}, \\ L_{\beta\beta} &= \frac{\partial L_\beta}{\partial \alpha}. \end{aligned}$$

We consider a Finsler space with an $(\alpha,\beta)-$metric $L(\alpha,\beta)$. The space $R^n = (M^n, \alpha)$ is called the associated Riemannian space. Let to $\gamma_j{}^i{}_k(x)$ be the Christoffel symbols constructed from α and we denote the covariant differentiation with respect $\gamma_j{}^i{}_k(x)$ by '/'. We put $a^{ij} = (a_{ij})^{-1}$, and we use the symbols as follows:

$$2r_{ij} = b_{i/j} + b_{j/i}, \quad 2s_{ij} = b_{i/j} - b_{j/i},$$

$$s^i_j = a^{ih} s_{hj}, \quad s_j = b_i s^i_j, \quad b^i = a^{ih} b_h, \quad b^2 = b^i b_i.$$

The well known Matsumotos theorem [11] is stated as :

Theorem 2.1. A Finsler space (M^n, L) with an (α, β)-metric $L(\alpha, \beta)$ is projectively flat if and only if for any point of space M there exist local coordinate neighborhoods containing the point such that $\gamma_j{}^i{}_k$ satisfies:

$$(\gamma_0{}^i{}_0 - \gamma_{000} y^i/\alpha^2)/2 + (\alpha L_\beta/L_\alpha) s^i_0 \\ +(L_{\alpha\alpha}/L_\alpha)(C + \alpha r_{00}/2\beta)(\alpha^2 b^i/\beta - y^i) = 0, \tag{2.1}$$

where C is given by,

$$C + (\alpha^2 L_\beta/\beta L_\alpha) s_0 + (\alpha L_{\alpha\alpha}/\beta^2 L_\alpha)(\alpha^2 b^2 - \beta^2)(C + \alpha r_{00}/2\beta) = 0. \tag{2.2}$$

Equation (2.2) can be written as,

$$(C + \alpha r_{00}/2\beta)\{1 + (\alpha L_{\alpha\alpha}/\beta^2 L_\alpha)(\alpha^2 b^2 - \beta^2)\} - (\alpha/2\beta)\{r_{00} - (2\alpha L_\beta/L_\alpha) s_0\} \\ = 0,$$

that is,

$$C + \alpha r_{00}/2\beta = \frac{\alpha\beta(r_{00} L_\alpha - 2\alpha L_\beta s_0)}{2\{\beta^2 L_\alpha + \alpha L_{\alpha\alpha}(\alpha^2 b^2 - \beta^2)\}}. \tag{2.3}$$

3. Finsler Space with the Metric $\alpha + \frac{2\beta^2}{\alpha} - \frac{1}{3}\frac{\beta^4}{\alpha^3} + \epsilon\beta$

In this section, we consider the Finsler space F^n with an (α, β)- metric L given by,

$$L = \alpha + \frac{2\beta^2}{\alpha} - \frac{1}{3}\frac{\beta^4}{\alpha^3} + \epsilon\beta. \tag{3.1}$$

The partial derivatives with respect to α and β of a metric (3.1) are given by,

$$L_\alpha = \frac{\alpha^4 - 2\alpha^2\beta^2 + \beta^4}{\alpha^4}, \; L_\beta = \frac{12\alpha^2\beta - 4\beta^3 + 3\alpha^3\epsilon}{3\alpha^3}, \; L_{\alpha\alpha} = \frac{4\beta^2(\alpha^2 - \beta^2)}{\alpha^5}. \tag{3.2}$$

Substituting (3.2) in (2.3) we get,

$$C + \alpha r_{00}/2\beta = \frac{\alpha\{r_{00}(3\alpha^4 - 2\alpha^2\beta^2 + \beta^4) - 2\alpha^2(12\alpha^2\beta - 4\beta^3 + 3\alpha^2\epsilon)s_0\}}{6\beta\{(\alpha^4 - 2\alpha^2\beta^2 + \beta^4) + (\alpha^2 - \beta^2)(\alpha^2 b^2 - \beta^2)\}}. \tag{3.3}$$

Plugging (3.2) and (3.3) in (2.1) we obtain,

$$(\gamma_0{}^i{}_0\alpha^2 - \gamma_{000} y^i)(\alpha^4 - 2\alpha^2\beta^2 + \beta^4)\{3(\alpha^4 - 2\alpha^2\beta^2 + \beta^4) \\ +3(\alpha^2 - \beta^2)(\alpha^2 b^2 - \beta^2)\} + 2\alpha^4(12\alpha^2\beta - 4\beta^3 + 3\alpha^3\epsilon)\{3(\alpha^4 - 2\alpha^2\beta^2 + \beta^4) \\ +3(\alpha^2 - \beta^2)(\alpha^2 b^2 - \beta^2)\} s^i_0 + 4\alpha^2(\alpha^2 - \beta^2)\{r_{00}(3\alpha^4 - 2\alpha^2\beta^2 + \beta^4) \\ -2\alpha^2(12\alpha^2\beta - 4\beta^3 + 3\alpha^2\epsilon)s_0\}(\alpha^2 b^i - y^i\beta) = 0. \tag{3.4}$$

Only the term $-6\beta^8\gamma_{000}y^i$ of (3.4) seemingly does not contain α^2, hence we must have hp(10)v^i_{10} satisfying $-6\beta^8\gamma_{000}y^i = \alpha^2 v^i_{10}$. For the sake of brevity we suppose $\alpha^2 \not\equiv 0(mod\beta)$. Then the above is written as,

$$\gamma_{000} = v_0\alpha^2, \tag{3.5}$$

where v_0 is hp(1). Substituting (3.5) in (3.4), we obtain

$$\begin{aligned}&(\gamma_0{}^i{}_0 - v_0y^i)(\alpha^4 - 2\alpha^2\beta^2 + \beta^4)\{3(\alpha^4 - 2\alpha^2\beta^2 + \beta^4)\\&+3(\alpha^2 - \beta^2)(\alpha^2b^2 - \beta^2)\} + 2\alpha^2(12\alpha^2\beta - 4\beta^3 + 3\alpha^3\epsilon)\{3(\alpha^4 - 2\alpha^2\beta^2 + \beta^4)\\&+3(\alpha^2 - \beta^2)(\alpha^2b^2 - \beta^2)\}s^i_0 + 4(\alpha^2 - \beta^2)\{r_{00}(3\alpha^4 - 2\alpha^2\beta^2 + \beta^4)\\&-2\alpha^2(12\alpha^2\beta - 4\beta^3 + 3\alpha^2\epsilon)s_0\}(\alpha^2b^i - y^i\beta) = 0.\end{aligned} \tag{3.6}$$

The terms $\beta^7\{6\beta(\gamma_0{}^i{}_0 - v_0y^i) + 4r_{00}y^i\}$ of (3.6) seemingly does not contain α^2. Consequently we must have hp(1)u^i_0 such that above is equal to $\alpha^2\beta^7u^i_0$, that is

$$6\beta(\gamma_0{}^i{}_0 - v_0y^i) + 4r_{00}y^i = \alpha^2u^i_0. \tag{3.7}$$

We contract (3.7) by $a_{ir}y^r$, which yields

$$4r_{00} = u^i_0y_i. \tag{3.8}$$

Now from (3.7) and (3.8), we obtain

$$\gamma_0{}^i{}_0 = v_0y_i, \tag{3.9}$$

which implies

$$2\gamma_j{}^i{}_k = v_k\delta^i_j + v_j\delta^i_k. \tag{3.10}$$

which shows that the associated Riemannian space is projectively flat.
Substituting (3.9) in (3.6), we obtain

$$\begin{aligned}&2\alpha^2(12\alpha^2\beta - 4\beta^3 + 3\alpha^3\epsilon)\{3(\alpha^4 - 2\alpha^2\beta^2 + \beta^4) + 3(\alpha^2 - \beta^2)(\alpha^2b^2 - \beta^2)\}s^i_0\\&+4(\alpha^2 - \beta^2)\{r_{00}(3\alpha^4 - 2\alpha^2\beta^2 + \beta^4) - 2\alpha^2(12\alpha^2\beta - 4\beta^3 + 3\alpha^2\epsilon)s_0\}\\&(\alpha^2b^i - y^i\beta) = 0.\end{aligned} \tag{3.11}$$

Transvecting (3.11) by b_i, we get

$$\begin{aligned}&72\alpha^6\beta(\alpha^2b^2 - 3\beta^2)s_0 - 72\alpha^4\beta^3(\alpha^2b^2 - 2\beta^2)s_0 - 24\alpha^4\beta^3(\alpha^2b^2 - 3\beta^2)s_0\\&+24\alpha^2\beta^5(\alpha^2b^2 - 2\beta^2)s_0 + 18\alpha^7(\alpha^2b^2 - 3\beta^2)\epsilon s_0 - 18\alpha^5\beta^2(\alpha^2b^2 - 2\beta^2)\epsilon s_0\\&+72\alpha^8\beta s_0 - 24\alpha^6\beta^3s_0 + 18\alpha^9\epsilon s_0 - 96(\alpha^2 - \beta^2)(\alpha^2b^2 - \beta^2)\alpha^4\beta s_0\\&+32(\alpha^2 - \beta^2)(\alpha^2b^2 - \beta^2)\alpha^2\beta^3s_0 - 24(\alpha^2 - \beta^2)(\alpha^2b^2 - \beta^2)\alpha^4\epsilon s_0\\&+4(\alpha^2 - \beta^2)(\alpha^2b^2 - \beta^2)(3\alpha^4 - 2\alpha^2\beta^2 + \beta^4)r_{00} = 0\end{aligned} \tag{3.12}$$

Since, $4(\alpha^2-\beta^2)(\alpha^2b^2-\beta^2)(3\alpha^4-2\alpha^2\beta^2+\beta^4)$ of (3.12) does not contain α^2 as a factor, we must have a function $k(x)$ such that,

$$r_{00}=k(x)\alpha^2. \tag{3.13}$$

Substituting (3.13) in (3.12), we obtain

$$\begin{aligned}
&72\alpha^4\beta(\alpha^2b^2-3\beta^2)s_0-72\alpha^2\beta^3(\alpha^2b^2-2\beta^2)s_0-24\alpha^2\beta^3(\alpha^2b^2-3\beta^2)s_0\\
&+24\beta^5(\alpha^2b^2-2\beta^2)s_0+18\alpha^5(\alpha^2b^2-3\beta^2)\epsilon s_0-18\alpha^3\beta^2(\alpha^2b^2-2\beta^2)\epsilon s_0\\
&+72\alpha^6\beta s_0-24\alpha^4\beta^3s_0+18\alpha^7\epsilon s_0-96(\alpha^2-\beta^2)(\alpha^2b^2-\beta^2)\alpha^2\beta s_0\\
&+32(\alpha^2-\beta^2)(\alpha^2b^2-\beta^2)\beta^3s_0-24(\alpha^2-\beta^2)(\alpha^2b^2-\beta^2)\alpha^2\epsilon s_0\\
&+4(\alpha^2-\beta^2)(\alpha^2b^2-\beta^2)(3\alpha^4-2\alpha^2\beta^2+\beta^4)k(x)=0,
\end{aligned} \tag{3.14}$$

leads to $k(x)=0$ because $(\alpha^2-\beta^2)(\alpha^2b^2-\beta^2)(3\alpha^4-2\alpha^2\beta^2+\beta^4)$ does not vanish. Hence we have,

$$r_{00}=0;\ \ r_{ij}=0\ \ \textit{and}\ \ s_0=0;\ \ s_i=0. \tag{3.15}$$

Plugging (3.14) in (3.11), we get $s^i_0=0$ that is $s_{ij}=0$
Since, $r_{ij}=0$ and $s_{ij}=0$, $b_{i/j}=0$. Conversely, if $b_{i/j}=0$, then we get $r_{00}=0, s^i_0=0$ and $s_0=0$. So (3.6) follows from (3.9).
Hence we conclude that

Theorem 3.1. A Finsler space F^n with an special (α,β)-metric given by (3.1) is projectively flat, if and only if we have $b_{i/j}=0$ and the associated Riemannian space (M^n,α) is projectively flat.

4. Conclusion

One of the fundamental problems in Finsler geometry is to study and characterize the projectively flat metrics on an open domain $U\epsilon R^n$. Projectively flat metrics on U are Finsler metrics whose geodesics are straight lines. The real starting point of the investigations of projectively flat metrics is Hilberts fourth problem [2] raised on the International Congress of Mathematicians (Paris 1900), in which he asked about the spaces in which the shortest curves between any pair of points are straight line segments. The first answer was given by Hilberts student G. Hamel. In a 34 page long paper he found necessary and sufficient conditions in order that a space satisfying an axiom system, which is a modification of Hilberts system of axioms for Euclidean geometry, removing a strong congruence axiom and including the Desargues axiom, be projectively flat.

Particularly, In this paper we have considered and proved that it is projectively flat if and only if the associated Riemannian space M^n is also projectively flat.

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