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On Bayesian Estimation of Maxwell Distribution under Precautionary Loss Function

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Abstract

In this paper Bayes estimator of the scale parameter of Maxwell distribution has been obtained by taking quasi, inverted gamma and uniform prior distributions using precautionary loss function. These estimators are compared with the corresponding Bayes estimators under squared error loss function.

Keywords and Phrases : Squared error loss function, prior distribution, precautionary loss function, posterior pdf and expectation, inverted gamma distribution.

1. Introduction

Let us consider the Maxwell distribution whose probability density function is given by

$$f(x; \theta) = \frac{4}{\sqrt{\pi}} \frac{x^2}{\theta^{3/2}} e^{-x^2/\theta}; \quad \theta > 0, x > 0, \quad (1.1)$$

where θ is the scale parameter. The Maxwell distribution plays a very important role of a life time model as for increasing θ the distribution becomes flatter and the right tail increases Tyagi and Bhattacharya [6] have obtained the MVU estimator of reliability.

Let us suppose that n items are put to life test and terminate the experiment when $r(< n)$ items have failed. If $x_1, x_2, \dots, x_r$ denote the first ' r ' observations having a common density function as given above, the joint probability density function is given by

$$f(\underline{x}|\theta) = \frac{n!}{(n-r)!} \left(\frac{4}{\sqrt{\pi}} \right)^r \left(\frac{1}{\theta} \right)^{3r/2} \left(\prod_{i=1}^r x_i^2 \right) e^{-(T_r/\theta)} \quad (1.2)$$

where $T_r = \left[\sum_{i=1}^r x_i^2 + (n-r) x_{(r)}^2 \right]$. Thus, the maximum likelihood estimator (MLE) θ is given by

$$\hat{\theta} = \frac{2T_r}{3r}. \quad (1.3)$$

The pdf of $\hat{\theta}$ is given by

$$f(\hat{\theta}) = \frac{\left(\frac{3r}{2\hat{\theta}}\right)^r}{\Gamma(3r/2)} \left(\hat{\theta}\right)^{\frac{3r}{2}-1} e^{-3r\hat{\theta}/2\theta}; \quad \hat{\theta} > 0. \quad (1.4)$$

The Bayes estimator $\hat{\theta}_L$ of θ is of course, optimal relative to the loss function chosen. A commonly used loss function is the squared error loss function (SELF)

$$L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2. \quad (1.5)$$

It is well known that the Bayes estimator under the above loss function, say $\hat{\theta}_S$, is the posterior mean. The squared error loss function (SELF) is often used due to the fact that it is symmetrical and also it does not lead to complicated numerical computation. Several authors (Ferguson [3], Varian [7], Berger [2], Zellner [8] and Basu and Ebrahimi [1], have recognized that the inappropriateness of using symmetric loss function in several estimation problems and proposed different asymmetric loss functions e.g., Linex and many variant forms of it.

Norstrom [4] introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss function with quadratic loss function as a special case. These loss functions approach infinitely near the origin to prevent underestimation and thus giving a conservative estimators, especially when low failure rates are being estimated. These estimators are very useful when underestimation may lead to serious consequences. A very useful and simple asymmetric precautionary loss function is

$$L(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\hat{\theta}}, \quad (1.6)$$

Bayes estimator of θ under precautionary loss function is denoted by $\hat{\theta}_P$ and may be obtained by solving the following equation

$$\hat{\theta}_P = [E_{\pi}(\theta^2)]^{\frac{1}{2}}, \quad (1.7)$$

where $E_{\pi}(\theta^2)$ is the posterior expectation of θ^2 .

In this paper, we have obtained Bayes estimator of θ using precautionary loss function, under three prior distribution, viz., quasi-density as given by

$$g_1(\theta) = \frac{1}{\theta^d}; \quad \theta > 0, d > 0 \quad (1.8)$$

here $d = 0$ leads to a diffuse prior and $d = 1$, a non-informative prior; the inverted gamma distribution with parameters α and $\beta(> 0)$ with p.d.f. given by

$$g_2(\theta) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} e^{-\beta/\theta} & ; \quad \theta > 0 \quad (\alpha, \beta) > 0 \\ 0 & ; \quad \text{otherwise} \end{cases} \quad (1.9)$$

The main reason for general acceptability is the mathematical tractability resulting from the fact that the inverted gamma distribution is conjugate prior for θ . The uniform distribution over $[\alpha, \beta]$ as

$$g_3(\theta) = \begin{cases} \frac{1}{\beta-\alpha} & ; \quad 0 < \alpha \leq \theta \leq \beta \\ 0 & ; \quad \text{otherwise} \end{cases} \quad (1.10)$$

2. Bayes Estimator under $g_1(\theta)$

The posterior pdf of θ under the prior $g_1(\theta)$, may be obtained, using equation (1.2)

$$f(\theta|\underline{x}) = \frac{T_r^{\frac{1}{2}(3r+2d-2)}}{\Gamma_{\frac{1}{2}}(3r+2d-2)} \theta^{-\frac{1}{2}(3r+2d)} e^{-(T_r/\theta)}; \quad \theta > 0, \quad r+d > 1. \quad (2.1)$$

The Bayes estimator under squared error loss function (SELF) is given by

$$\hat{\theta}_S = \frac{2T_r}{(3r+2d-4)}. \quad (2.2)$$

Also, the Bayes estimator under precautionary loss function, say $\hat{\theta}_P$ is obtained as

$$\hat{\theta}_P = \frac{2T_r}{[(3r+2d-4)(3r+2d-6)]^{1/2}}. \quad (2.3)$$

The Risk Functions

The risk function of the estimators $\hat{\theta}_S$ and $\hat{\theta}_P$, relative to squared error loss function are denoted by $R_S(\hat{\theta}_S)$ and $R_S(\hat{\theta}_P)$, respectively and those relative to precautionary loss $R_P(\hat{\theta}_S)$ and $R_P(\hat{\theta}_P)$, respectively are given by

$$R_S(\hat{\theta}_S) = \theta^2 \left[\frac{3r(3r+2)}{(3r+2d-2)^2} - \frac{6r}{(3r+2d-4)} + 1 \right], \quad (2.4)$$

$$R_S(\hat{\theta}_P) = \theta^2 \left[\frac{3r(3r+2)}{[(3r+2d-4)(3r+2d-6)]} - \frac{6r}{[(3r+2d-4)(3r+2d-6)]^{1/2}} + 1 \right], \quad (2.5)$$

$$R_P(\hat{\theta}_S) = \theta \left[\frac{3r+2d-4}{3r-2} + \frac{3r}{3r+2d-4} - 2 \right], \quad (2.6)$$

$$R_P(\hat{\theta}_P) = \theta \left[\frac{[(3r+2d-4)(3r+2d-6)]^{1/2}}{3r-2} + \frac{3r}{[(3r+2d-4)(3r+2d-6)]^{1/2}} - 2 \right]. \quad (2.7)$$

3. Bayes Estimator under $g_2(\theta)$

The posterior pdf of θ under the prior $g_2(\theta)$, may be obtained, using equation (1.2) comes out to be

$$f(\theta | \underline{x}) = \frac{(\beta + T_r)^{(3r+2\alpha)/2}}{\Gamma_{\frac{1}{2}}(3r+2\alpha)} \theta^{-(3r+2\alpha+2)/2} e^{-\frac{1}{\theta}(\beta+T_r)}. \quad (3.1)$$

Using equation (3.1), the Bayes estimator under SELF, say $\hat{\theta}_S$, is given by

$$\hat{\theta}_S = \frac{2(\beta + T_r)}{(3r + 2\alpha - 2)}. \quad (3.2)$$

Equations (1.7) and (3.1) lead to the Bayes estimator, say $\hat{\theta}_P$, under precautionary loss function is given by

$$\hat{\theta}_P = \frac{2(\beta + T_r)}{[(3r + 2\alpha - 2)(3r + 2\alpha - 4)]^{1/2}}. \quad (3.3)$$

The Risk Functions

The risk function of the estimators $\hat{\theta}_S$ and $\hat{\theta}_P$, relative to squared error loss function are given by

$$R_S(\hat{\theta}_S) = \theta^2 \left[\left(\frac{3r(3r+2) + \frac{12r\beta}{\theta} + \frac{4\beta^2}{\theta^2}}{(3r+2\alpha-2)^2} \right) - \frac{2 \left(3r + \frac{2\beta}{\theta} \right)}{(3r+2\alpha-2)} + 1 \right], \quad (3.4)$$

and

$$R_S(\hat{\theta}_P) = \theta^2 \left[K^2 \left(\frac{3r}{4}(3r+2) + 3r \left(\frac{\beta}{\theta} \right) + \left(\frac{\beta}{\theta} \right)^2 \right) - K \left(3r + \left(\frac{2\beta}{\theta} \right) \right) + 1 \right], \quad (3.5)$$

where

$$K = 2[(3r+2\alpha-2)(3r+2\alpha-4)]^{-1/2}.$$

The risk function and Bayes risk of the estimator $\hat{\theta}_S$ and $\hat{\theta}_P$, relative to precautionary loss are not obtainable in closed form.

4. Bayes Estimator under $g_3(\theta)$

The posterior pdf of θ under the prior $g_3(\theta)$, may be obtained as

$$f(\theta|\underline{x}) = \frac{T_r^{(3r-2)/2} \theta^{-3r/2} e^{-T_r/\theta}}{I_g\left(\frac{T_r}{\alpha}, \frac{3r}{2} - 1\right) - I_g\left(\frac{T_r}{\beta}, \frac{3r}{2} - 1\right)}, \quad (4.1)$$

where $I_g(x, n) = \int_0^x e^{-t} t^{n-1} dt$ is the incomplete gamma function.

The Bayes estimator under squared error loss function is given by

$$\hat{\theta}_S = \left(\frac{I_g\left(\frac{T_r}{\alpha}, \frac{3r}{2} - 2\right) - I_g\left(\frac{T_r}{\beta}, \frac{3r}{2} - 2\right)}{I_g\left(\frac{T_r}{\alpha}, \frac{3r}{2} - 1\right) - I_g\left(\frac{T_r}{\beta}, \frac{3r}{2} - 1\right)} \right) T_r. \quad (4.2)$$

Equation (1.7) on using equation (4.1) yields, the Bayes estimator under precautionary loss function as given by

$$\hat{\theta}_P = \left[\frac{I_g\left(\frac{T_r}{\alpha}, \frac{3r}{2} - 3\right) - I_g\left(\frac{T_r}{\beta}, \frac{3r}{2} - 3\right)}{I_g\left(\frac{T_r}{\alpha}, \frac{3r}{2} - 1\right) - I_g\left(\frac{T_r}{\beta}, \frac{3r}{2} - 1\right)} \right]^{\frac{1}{2}} T_r. \quad (4.3)$$

In this case risk functions and Bayes risks cannot be obtained in a closed form.

5. Conclusion

It is evident from the equations (2.2), (2.3), (3.2), (3.3), (4.2) and (4.3) that Bayes estimators of the shape parameter of the Maxwell distribution, under squared error and precautionary loss functions using quasi, natural conjugate and uniform priors, have different expressions for their definitions. The Bayes estimators do depend upon the parameters of the prior distributions.

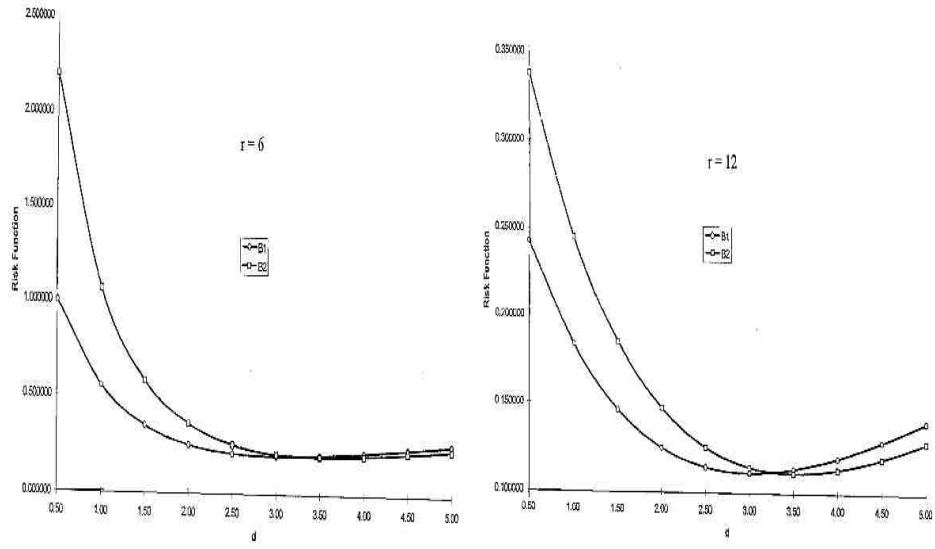


Fig. 1. Comparative Graph of Risk Function under SELF

In figure-1 we have plotted the risk functions B_1 and B_2 of the Bayes estimators $\hat{\theta}_S$ and $\hat{\theta}_P$, respectively, under squared error loss function, as given in equation (2.4) and (2.5) for $a = 1$, $r = 6(6)12$ and $d = 0.5(0.5)5.0$.

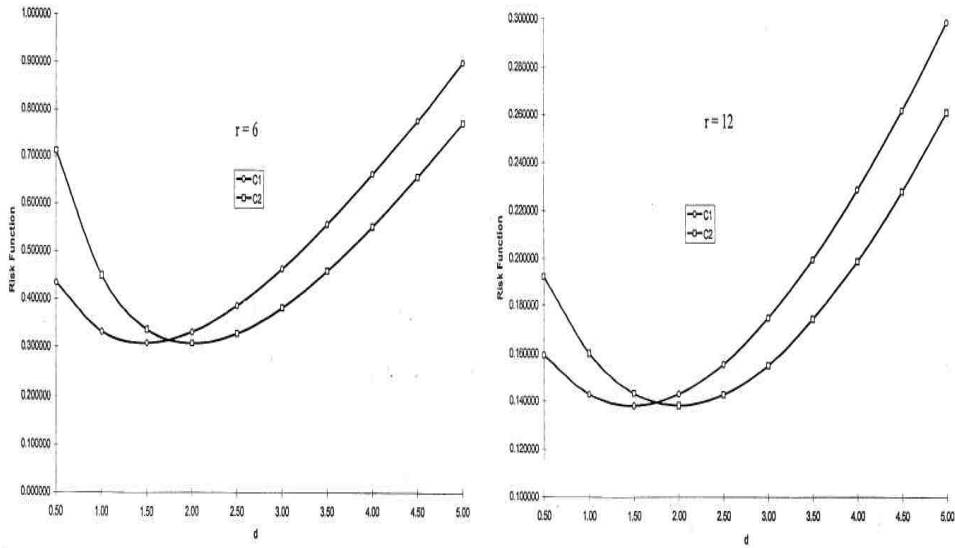


Fig. 2. Comparative Graph of Risk Function under Precautionary Loss Function

In figure-2 we have plotted the risk functions C_1 and C_2 of the Bayes estimators $\hat{\theta}_S$ and $\hat{\theta}_P$, respectively, under precautionary loss function, as given in equation (2.6) and (2.7) for $a = 1$, $r = 6(6)12$ and $d = 0.5(0.5)5.0$.

From figure-1 and figure-2, it is clear that neither of the estimators uniformly dominates the other.

The risks $R_S(\hat{\theta}_S)$ and $R_S(\hat{\theta}_P)$ may be calculated with the help of equation (3.4) and (3.5) for the given values of r , β and θ and the corresponding comparisons may be made under the prior distribution $g_2(\theta)$. The risks $R_P(\hat{\theta}_S)$ and $R_P(\hat{\theta}_P)$ are not obtainable in closed forms. Thus only numerical methods may be employed to obtain the values of the risks.

Under the prior $g_2(\theta)$ only simulation study may be made to compare the estimators under the two loss functions.

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