

J. T. S.

Vol. 9 (2015), pp.9-26

<https://doi.org/10.56424/jts.v9i01.10570>

A Self-similar Solution of a Shock Wave Propagation in a Perfectly Conducting Dusty Gas

J. P. Vishwakarma and Prem Lata

Department of Mathematics and Statistics,

D.D.U. Gorakhpur University, Gorakhpur 273009, India

E-mail: padma2012gkp@gmail.com and jpv_univgkp@yahoo.com

(Received: July 11, 2015)

Abstract

Self-similar solutions are obtained for unsteady, one-dimensional adiabatic (or isothermal) flow behind a strong shock in a perfectly conducting dusty gas in presence of a magnetic field. The shock wave is driven out by a piston moving with time according to power law. The initial magnetic field varies as some power of distance and the initial density of the medium is constant. The dusty gas is taken as the mixture of a perfect gas and small solid particles. It is assumed that the equilibrium flow condition is maintained in the flow field, and that the viscous-stress and heat conduction of the mixture are negligible. Solutions are obtained, in both cases, when the flow between the shock and the piston is isothermal or adiabatic. Effects of a change in the mass concentration of the solid particles in the mixture k_p , in the ratio of the density of solid particles to the initial density of the gas G_0 and in the strength of initial magnetic field are also obtained. It is shown that the presence of magnetic field has decaying effect on the shock wave, but this effect is decreased on increasing k_p when $G_0 = 1$. Also, a comparison is made between adiabatic and isothermal cases.

Keywords : Shock wave, self-similar solution, dusty gas, magnetic field, adiabatic flow and isothermal flow.

2010 AMS Subject Classification : 76L05, 35L67.

1. Introduction

The study of shock wave in a mixture of small solid particles and perfect gas is of great interest in several branches of engineering and science (Pai et al. [20]). The dust phase constitutes the total amount of solid particles which are continuously distributed in perfect gas. The volumetric fraction of the dust

lowers the compressibility of the mixture, and the mass of the dust load may increase the total mass, and hence it may add to the inertia of the mixture. Both effects due to addition of the dust, the decrease of the mixture's compressibility and the increase of the mixture's inertia may markedly influence the shockwave.

Miura and Glass [16] obtained an analytic solution for a planar dusty gas flow with constant velocities of the shock and piston moving behind it. As they neglected the volume occupied by the solid particles mixed into the perfect gas, the dust virtually has a mass fraction but no volume fraction. Their results reflect the influence of the additional inertia of the dust upon the shock propagation. For plane, cylindrical and spherical geometry Vishwakarma [28] computed a non-similarity solution for the flow field behind a strong shock propagating at non-constant velocity in a dusty gas. He considered exponential time dependence for the velocity of the shock. As he considered the nonzero volume fraction of solid particles in dusty gas, his results reflect the effect of both the decrease of compressibility and the increase of the inertia of the medium on the shock propagation (Steiner and Hirschler [26], Vishwakarma and Pandey [30]). The similarity method of Taylor [27] and Sedov [24] well known for piston problems have been used by several authors, e.g. Finkleman and Baron [6], Gretler and Regenfelter [9], Helliwell [11], Wang [34], Singh et al. [25], to discuss about the hyperbolic character of the governing equations and to obtain solutions in an ideal gas. Steiner and Hirschler [26] have derived similarity solutions for the flow behind a shock wave propagating in a dusty gas. The shock wave is driven out by a moving piston with time according to power law.

At high temperatures that prevail in the problems associated with shock waves a gas is ionized and electromagnetic effects may also be significant. A complete analysis of such a problem should therefore consist of the study of gas dynamic flow and the electromagnetic field simultaneously. The study of propagation of cylindrical shock waves in a conducting gas in the presence of an axial or azimuthal magnetic field is relevant to the experiments on pinch effect, exploding wires, and so on. This problem both in uniform and non-uniform ideal gas was under taken by many investigators such as Pai [18], Sakurai [23], Bhutani [2], Cole and Greifinger [4], DebRay [5], Christer and Helliwell [3], Vishwakarma and Yadav [33], Vishwakarma and Patel [31]. Vishwakarma and Singh [32] have studied the propagation of diverging shock waves in a low conducting and uniform or non-uniform gas as a result of time dependent energy input [31, 14] under the influence of a spatially variable axial magnetic induction. Vishwakarma et al. [7] have extended the work of Vishwakarma and Singh

[32] to study the propagation of diverging cylindrical shock waves in a weakly conducting dusty gas in place of a perfect gas.

The magnetic fields have important roles in a variety of astrophysical situations. Complex filamentary structure in molecular clouds, shapes and the shaping of planetary nebulae, synchrotron radiation from supernova remnants, magnetized stellar winds, galaxies, and galaxy clusters as well as other interesting problems all involve magnetic fields (see [17,10,1]). In the present paper, we generalize the solution given by Steiner and Hirschler [26] for the propagation of a strong shock wave in a conducting dusty gas in presence of a magnetic field driven out by a piston moving according to a power law. The initial magnetic field varies as some power of distance and the initial density of the medium is constant. In order to get some essential features of shock propagation in the presence of a magnetic field, the solid particles are considered as a pseudo-fluid continuously distributed in the perfect gas and the mixture as perfectly conducting fluid. It is also assumed that the equilibrium flow condition is maintained in the flow field, and that the viscous stress and heat conduction of the mixture are negligible (Pai et al.[20], Higashino and Suzuki [12]). In this paper, both the adiabatic and isothermal flows between the shock and the piston are considered. The assumption of adiabaticity may not be valid for the high temperature flow where the intense heat transfer takes place such as behind a strong shock. Therefore, an alternative assumption of zero-temperature gradient throughout the flow (isothermal flow) may approximately be taken (Korobeinikov [13], Laumbach and Probst [14], Sachdev and Ashraf [22]). The effects of variation of mass concentration of solid particles (k_p), the ratio of density of solid particles to the initial density of the perfect gas in the mixture (G_0) and the parameter for strength of initial magnetic field (M_A^{-2}) are obtained. A comparative study between the solutions of isothermal and adiabatic flows is also made.

2. Fundamental Equations and Boundary Conditions: Adiabatic Flow

The fundamental equations for one-dimensional, unsteady and adiabatic flow of a perfectly conducting mixture of a gas and small solid particles in the presence of an azimuthal magnetic field may be written as (c.f. Pai et al. [20], Whitham [35])

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \rho \frac{\partial u}{\partial r} + j \frac{u \rho}{r} = 0, \quad (2.1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial h}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{1}{\rho} \left[\mu h \frac{\partial h}{\partial r} + \frac{\mu h^2}{r} \right] = 0, \quad (2.2)$$

$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial r} + h \frac{\partial u}{\partial r} + (j-1) \frac{hu}{r} = 0, \quad (2.3)$$

$$\frac{\partial e}{\partial t} + u \frac{\partial e}{\partial r} - \frac{p}{\rho^2} \left[\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} \right] = 0, \quad (2.4)$$

where ρ is the density, u is the flow velocity, p is the pressure, h is the azimuthal magnetic field, e is the internal energy per unit mass, μ is the magnetic permeability, r and t are the space and the time coordinates respectively and $j = 1, 2$ correspond to the cylindrical and the spherical symmetries.

The equation of state of the mixture of a perfect gas and small solid particles can be written as (Pai [19])

$$p = \frac{1 - k_p}{1 - Z} \rho R^* T, \quad (2.5)$$

where R^* is the gas constant, k_p the mass concentration of the solid particles, T the temperature and Z the volume fraction of the solid particles in the mixture.

The relation between k_p and Z is given by

$$k_p = \frac{Z \rho_{sp}}{\rho}, \quad (2.6)$$

where ρ_{sp} is species density of solid particles.

In the equilibrium flow, k_p is a constant in the whole flow-field. Therefore

$$\frac{Z}{\rho} = \text{constant}. \quad (2.7)$$

Also we have the relation

$$Z = \frac{k_p}{(1 - k_p)G + k_p}, \quad (2.8)$$

where $G = \frac{\rho_{sp}}{\rho}$ is the ratio of the density of the solid particles to the density of the perfect gas in the mixture.

The Internal energy per unit mass of the mixture may be written as

$$e = [k_p C_{sp} + (1 - k_p) C_v] T = C_{vm} T, \quad (2.9)$$

where C_{sp} is the specific heat of solid particles, C_v the specific heat of the gas at constant volume and C_{vm} the specific heat of the mixture at constant volume process.

The specific heat of the mixture at constant pressure is

$$C_{pm} = k_p C_{sp} + (1 - k_p) C_p, \quad (2.10)$$

where C_p is a specific heat of the gas at constant pressure.

The ratio of the specific heats of the mixture is given by (Pai [19], Marble [15])

$$\Gamma = \frac{C_{pm}}{C_{vm}} = \frac{1 + \frac{\delta\beta'}{\gamma}}{1 + \delta\beta'}, \quad (2.11)$$

where $\gamma = \frac{C_p}{C_v}$, $\delta = \frac{k_p}{1-k_p}$ and $\beta' = \frac{C_{sp}}{C_v}$.

Now,

$$C_{pm} - C_{vm} = (1 - k_p)(C_p - C_v) = (1 - k_p)R^*. \quad (2.12)$$

The internal energy per unit mass of the mixture is, therefore, given by

$$e = \frac{p(1 - Z)}{\rho(\Gamma - 1)}. \quad (2.13)$$

The equilibrium speed of sound in the mixture 'a' is given by

$$a^2 = \frac{\Gamma p}{\rho(1 - Z)}. \quad (2.14)$$

A strong cylindrical or spherical shock is supposed to be propagating in the undisturbed electrically conducting mixture of an ideal gas and small solid particles with constant density.

The azimuthal magnetic field in undisturbed dusty gas is assumed to vary as

$$h = \frac{A}{r^l}, \quad (2.15)$$

where 'A' and 'l' are constants. The flow variables immediately ahead of the shock front are

$$u = u_0 = 0, \quad (2.16)$$

$$\rho = \rho_0 = \text{constant}, \quad (2.17)$$

$$h = h_0 = Ar_s^{-l}, \quad (2.18)$$

$$p = p_0 = \frac{(1 - l)\mu A^2}{2lr_s^{2l}}, \quad (0 < l < 1), \quad (2.19)$$

where r_s is the shock radius and subscript '0' denotes the conditions immediately ahead of the shock.

The laws of conservation of mass, magnetic flux, momentum and energy across the shock front propagating with velocity $U_s (= \frac{dr_s}{dt})$ into a medium (mixture of an ideal gas and small solid particles) of constant density ρ_0 at rest

($u_0 = 0$) and with negligibly small counter pressure $p_0 \cong 0$ give the following shock conditions:

$$\rho_0 U_s = \rho_s (U_s - u_s), \quad (2.20)$$

$$h_0 U_s = h_s (U_s - u_s), \quad (2.21)$$

$$\frac{1}{2} \mu h_0^2 + \rho_0 U_s^2 = p_s + \rho_s (U_s - u_s)^2 + \frac{1}{2} \mu h_s^2, \quad (2.22)$$

$$\frac{U_s^2}{2} + \frac{\mu h_0^2}{\rho_0} = e_s + \frac{p_s}{\rho_s} + \frac{(U_s - u_s)^2}{2} + \frac{\mu h_s^2}{\rho_s}, \quad (2.23)$$

$$\frac{Z_s}{\rho_s} = \frac{Z_0}{\rho_0}, \quad (2.24)$$

where the subscript ‘s’ denotes conditions immediately behind the shock front.

The shock conditions (2.20-2.23) reduce to

$$\rho_s = \frac{\rho_0}{\beta}, \quad (2.25a)$$

$$h_s = \frac{h_0}{\beta}, \quad (2.25b)$$

$$Z_s = \frac{Z_0}{\beta}, \quad (2.25c)$$

$$u_s = (1 - \beta) U_s, \quad (2.25d)$$

$$p_s = \left[(1 - \beta) + \frac{1}{2M_A^2} \left(1 - \frac{1}{\beta^2} \right) \right] \rho_0 U_s^2, \quad (2.25e)$$

where β ($0 < \beta < 1$) is given by the relation

$$\beta^3(\Gamma + 1) - \beta^2\{(M_A^{-2} + 1)\Gamma + 2Z_0 - 1\} + \beta\{Z_0 + \Gamma - 2\}M_A^{-2} + Z_0M_A^{-2} = 0, \quad (2.26)$$

Z_0 being the initial volume fraction of the solid particles in the mixture and M_A the Alfven Mach number.

The expression for the initial volume fraction of the solid particles Z_0 is given by

$$Z_0 = \frac{k_p}{(1 - k_p)G_0 + k_p}, \quad (2.27)$$

where G_0 is the ratio of the density of solid particles to the initial density of the perfect gas. Also the Alfven Mach number M_A is given by

$$M_A^2 = \frac{U_s^2}{\frac{\mu h_0^2}{\rho_0}}. \quad (2.28)$$

3. Self-similarity Transformations

The flow field is bounded by a spherical (or cylindrical) piston internally and a spherical (or cylindrical) shock externally. In the framework of self-similarity (Sedov [7]) the velocity U_p of the piston is assumed to follow a power law given by

$$U_p = \frac{dr_p}{dt} = U_0 \left(\frac{t}{t_0} \right)^n, \quad (3.1)$$

where t_0 is the time at a reference state, r_p denotes the radius of the piston, U_0 is the piston velocity at $t = t_0$ and n is a constant. The consideration of ambient pressure p_0 and ambient magnetic field h_0 imposes restriction on ‘ n ’ ($-\frac{1}{2} < n < 0$) (see equation (3.6)). Thus the piston velocity jumps, almost instantaneously from zero to infinity leading to the formation of a shock of high strength in the initial phase. Referring the shock boundary conditions, self-similarity requires that the velocity of the shock U_s is proportional to the velocity of the piston, that is,

$$U_s = \frac{dr_s}{dt} = CU_0 \left(\frac{t}{t_0} \right)^n, \quad (3.2)$$

where C is a constant. The time and space coordinates can be transformed into a dimensionless self-similarity variable as follows

$$\lambda = \frac{r}{r_s} = \left[\frac{(n+1)t_0^n}{CU_0} \right] \left[\frac{r}{t^{n+1}} \right]. \quad (3.3)$$

Evidently, $\lambda = \lambda_p = \frac{r_p}{r_s}$ at the piston and $\lambda = 1$ at the shock.

To obtain the similarity solutions, we write the unknown variables in the following form (c.f. Steiner and Hirschler [4])

$$u = \phi(\lambda) \frac{r}{t}, \quad \rho = \Lambda(\lambda) \rho_0, \quad p = \psi(\lambda) \rho_0 \frac{r^2}{t^2}, \quad \mu^{1/2} h = \rho_0^{1/2} \frac{r}{t} \epsilon(\lambda), \quad Z = \Lambda(\lambda) Z_0, \quad (3.4)$$

where ϕ , Λ , ψ and ϵ are functions of λ only.

For existence of similarity solutions M_A should be a constant, therefore

$$m = \frac{n}{n+1}. \quad (3.5)$$

Since

$$0 < m < 1, \quad \left(-\frac{1}{2} < n < 0 \right). \quad (3.6)$$

The conservation equations (1.1) - (1.4) can be transformed into the following system of ordinary differential equations with respect to λ :

$$[\phi - (n+1)] \frac{d\Lambda}{d\lambda} + \Lambda \frac{d\phi}{d\lambda} = -\frac{\Lambda\phi(j+1)}{\lambda}, \quad (3.7)$$

$$[\phi - (n+1)] \frac{d\phi}{d\lambda} + \frac{1}{\Lambda} \frac{d\psi}{d\lambda} + \frac{\epsilon}{\Lambda} \frac{d\epsilon}{d\lambda} = -\frac{2(\psi + \epsilon^2)}{\Lambda\lambda} - \frac{(\phi^2 - \phi)}{\lambda}, \quad (3.8)$$

$$[\phi - (n+1)] \frac{d\epsilon}{d\lambda} + \epsilon \frac{d\phi}{d\lambda} = \frac{\epsilon - \phi\epsilon(j+1)}{\lambda}, \quad (3.9)$$

$$[\phi - (n+1)](1 - Z_0\Lambda) \frac{d\psi}{d\lambda} + \psi\Gamma \frac{d\phi}{d\lambda} = -\frac{2\psi(\phi - 1)(1 - Z)}{\lambda} - \frac{\phi\Gamma\psi(j+1)}{\lambda}, \quad (3.10)$$

By solving the above four equations, we get

$$\begin{aligned} \frac{d\phi}{d\lambda} = & \frac{(j+1)\phi\psi\Gamma - 2\psi(1-\phi)(1-Z_0\Lambda) + (j+1)(1-Z_0\Lambda)\phi\epsilon^2 - \epsilon^2(1-Z_0\Lambda)}{\lambda[\Lambda(1-Z_0\Lambda)(\phi-(n+1))^2 - \psi\Gamma - \epsilon^2(1-Z_0\Lambda)]} \\ & - \frac{\{2(\psi + \epsilon^2) + (\phi^2 - \phi)\Lambda\}\{\phi - (n+1)\}(1-Z_0\Lambda)}{\lambda[\Lambda(1-Z_0\Lambda)(\phi-(n+1))^2 - \psi\Gamma - \epsilon^2(1-Z_0\Lambda)]}, \end{aligned} \quad (3.11)$$

$$\begin{aligned} \frac{d\Lambda}{d\lambda} = & \frac{[2\psi(1-\phi) + \epsilon^2 - (j+1)\Lambda\phi\{\phi - (n+1)\}^2 + \{2(\psi + \epsilon^2) + (\phi^2 - \phi)\Lambda\}]}{\lambda[\Lambda(1-Z_0\Lambda)(\phi-(n+1))^2 - \psi\Gamma - \epsilon^2(1-Z_0\Lambda)](\phi - (n+1))} \\ & \frac{\{\phi - (n+1)\}\Lambda(1-Z_0\Lambda)}{\lambda[\Lambda(1-Z_0\Lambda)(\phi-(n+1))^2 - \psi\Gamma - \epsilon^2(1-Z_0\Lambda)](\phi - (n+1))}, \end{aligned} \quad (3.12)$$

$$\begin{aligned} \frac{d\epsilon}{d\lambda} = & \frac{\{1 - (j+1)\phi\}\{\phi - (n+1)\}^2(1-Z_0\Lambda)\Lambda\epsilon - \psi\Gamma\epsilon + 2\epsilon\psi(1-\phi)(1-Z_0\Lambda)}{\lambda[\Lambda(1-Z_0\Lambda)(\phi-(n+1))^2 - \psi\Gamma - \epsilon^2(1-Z_0\Lambda)](\phi - (n+1))} \\ & + \frac{\{2(\psi + \epsilon^2) + (\phi^2 - \phi)\Lambda\}\{\phi - (n+1)\}\epsilon(1-Z_0\Lambda)}{\lambda[\Lambda(1-Z_0\Lambda)(\phi-(n+1))^2 - \psi\Gamma - \epsilon^2(1-Z_0\Lambda)](\phi - (n+1))}, \end{aligned} \quad (3.13)$$

$$\begin{aligned} \frac{d\psi}{d\lambda} = & \frac{\{2(1-\phi)(1-Z_0\Lambda) - (j+1)\Gamma\phi\}\{\phi - (n+1)\}^2\Lambda\psi + \epsilon^2\psi\Gamma - 2\epsilon^2\psi(1-\phi)}{\lambda[\Lambda(1-Z_0\Lambda)(\phi-(n+1))^2 - \psi\Gamma - \epsilon^2(1-Z_0\Lambda)](\phi - (n+1))} \\ & \frac{(1-Z_0\Lambda) + \{2(\psi + \epsilon^2) + (\phi^2 - \phi)\Lambda\}\{\phi - (n+1)\}\psi\Gamma}{\lambda[\Lambda(1-Z_0\Lambda)(\phi-(n+1))^2 - \psi\Gamma - \epsilon^2(1-Z_0\Lambda)](\phi - (n+1))}. \end{aligned} \quad (3.14)$$

The piston's path coincides at $\lambda_p = \frac{r_p}{r_s}$ with a particle path. Using equations (3.1) and (3.4) the relation

$$\phi(\lambda_p) = (n+1), \quad (3.15)$$

can be derived.

Using the self-similarity transformations (3.4) and equation (3.2) the shock conditions (2.24) take the form

$$\phi(1) = (1 - \beta)(n+1), \quad (3.16a)$$

$$\Lambda(1) = \frac{1}{\beta}, \quad (3.16b)$$

$$\epsilon(1) = \frac{(n+1)}{\beta} M_A^{-2}, \quad (3.16c)$$

$$\psi(1) = \left[(1-\beta) + \frac{1}{2} M_A^{-2} \left(1 - \frac{1}{\beta^2} \right) \right] (n+1)^2. \quad (3.16d)$$

Now the differential equations (3.11-3.14) maybe numerically integrated, with the boundary conditions (3.16) to obtain the flow-field between the shock front and the piston.

4. Isothermalflow

In this section, we present the similarity solution for the isothermal flow behind a strong shock driven out by a spherical (or cylindrical) piston moving according to the powerlaw (3.1), in the case of perfectly conducting dusty gas.

The strong shock conditions, which serve as the boundary conditions for the problem will be same as the shock conditions (2.20-2.23) in the case of adiabatic flow.

For isothermal flow, equation (2.4) is replaced by

$$\frac{\partial T}{\partial r} = 0. \quad (4.1)$$

The equations (2.1), (2.2), and (2.3) can be transformed using equation (3.4) into

$$[\phi - (n+1)] \frac{d\Lambda}{d\lambda} + \Lambda \frac{d\phi}{d\lambda} = -\frac{\Lambda \phi (j+1)}{\lambda}, \quad (4.2)$$

$$[\phi - (n+1)] \frac{d\phi}{d\lambda} + H \frac{d\Lambda}{d\lambda} + \frac{\epsilon}{\Lambda} \frac{d\epsilon}{d\lambda} = -\frac{2\epsilon^2}{\Lambda\lambda} - \frac{(\phi^2 - \phi)}{\lambda}, \quad (4.3)$$

$$[\phi - (n+1)] \frac{d\epsilon}{d\lambda} + \epsilon \frac{d\phi}{d\lambda} = \frac{\epsilon - \phi \epsilon (j+1)}{\lambda}, \quad (4.4)$$

where

$$H = H(\lambda) = \frac{\psi(\lambda)}{\Lambda^2(1 - \Lambda Z_0)} = \frac{\left[(1-\beta) + \frac{1}{2} M_A^{-2} \left(1 - \frac{1}{\beta^2} \right) \right] (n+1)^2 (\beta - Z_0)}{\lambda^2 \Lambda (1 - \Lambda Z_0)^2}. \quad (4.5)$$

Equation (4.1) together with equation of state (2.5) gives

$$\frac{p}{p_s} = \frac{\rho}{\rho_s} \frac{(1 - Z_s)}{(1 - Z)}. \quad (4.6)$$

Equation (4.6) with the aid of equation (3.4) yields a relation between $\psi(\lambda)$ and $\Lambda(\lambda)$ in the form

$$\psi(\lambda) = \frac{\left[(1 - \beta) + \frac{1}{2} M_A^{-2} \left(1 - \frac{1}{\beta^2} \right) \right] (n+1)^2 (\beta - Z_0) \Lambda(\lambda)}{\lambda^2 (1 - \Lambda Z_0)}. \quad (4.7)$$

Solving equations (4.2)-(4.4) for $\frac{d\phi}{d\lambda}$, $\frac{d\epsilon}{d\lambda}$ and $\frac{d\Lambda}{d\lambda}$, we have

$$\frac{d\phi}{d\lambda} = \frac{\left\{ (\phi - \phi^2) - \frac{2\epsilon^2}{\Lambda} \right\} (\phi - (n+1)) + \left(H\Lambda + \frac{\epsilon^2}{\Lambda} \right) (j+1)\phi - \frac{\epsilon^2}{\Lambda}}{\lambda[(\phi - (n+1)^2) - H\Lambda - \frac{\epsilon^2}{\Lambda}]}, \quad (4.8)$$

$$\frac{d\epsilon}{d\lambda} = \frac{\epsilon[(1 - (j+1)\phi)(\phi - (n+1))^2 + (\phi - (n+1)) \left(\frac{2\epsilon^2}{\Lambda} - (\phi - \phi^2) \right) - H\Lambda]}{\lambda[(\phi - (n+1)^2) - H\Lambda - \frac{\epsilon^2}{\Lambda}](\phi - (n+1))}, \quad (4.9)$$

$$\frac{d\Lambda}{d\lambda} = - \frac{\Lambda \left[\phi(j+1)(\phi - (n+1))^2 + \left((\phi - \phi^2) - \frac{2\epsilon^2}{\Lambda} \right) (\phi - (n+1)) - \frac{\epsilon^2}{\Lambda} \right]}{\lambda[(\phi - (n+1)^2) - H\Lambda - \frac{\epsilon^2}{\Lambda}](\phi - (n+1))}, \quad (4.10)$$

where

$$H = H(\lambda) = \frac{\left[(1 - \beta) + \frac{1}{2} M_A^{-2} \left(1 - \frac{1}{\beta^2} \right) \right] (n+1)^2 (\beta - Z_0)}{\lambda^2 \Lambda (1 - \Lambda Z_0)^2}. \quad (4.11)$$

The transformed shock conditions (3.16) and the kinematic condition (3.15) at the piston will be same as in the case of adiabatic flow.

The ordinary differential equations (4.8-4.11) with boundary conditions (3.16) can now be numerically integrated to obtain the solution for the isothermal flow behind the shock surface. Normalizing the variables u , p , ρ and h with their respective values at the shock, we obtain

$$\frac{u}{u_s} = \frac{\phi}{\phi(1)} \lambda, \quad \frac{\rho}{\rho_s} = \frac{\Lambda}{\Lambda(1)}, \quad \frac{p}{p_s} = \frac{\psi}{\psi(1)} \lambda^2, \quad \frac{h}{h_s} = \frac{\epsilon}{\epsilon(1)} \lambda. \quad (4.12)$$

5. Results and Discussion

Equations (3.11-3.14) for adiabatic flow and equations (4.8-4.10) for isothermal flow with boundary conditions (3.16) were integrated using fourth-order Runge-kutta algorithm. The flow variables ϕ , Λ , ϵ and ψ as functions of λ are obtained from the shock front ($\lambda = 1$) until the inner expanding surface ($\lambda = \lambda_p$) is reached. For the purpose of numerical calculations, the values of constant parameters are taken to be (Pai et al. [20] Miura and Glass[16], Vishwakarma[28], Steiner and Hirschler[26], Rosenau and Frankenthal[21]) $j = 2$,

$\gamma = \frac{5}{3}$, $n = -0.15$, $\beta' = 0.25$, $k_p = 0, 0.2$, $G_0 = 1, 100$ and $M_A^{-2} = 0, 0.005, 0.01$. The value $j = 2$ corresponds to spherical shock, $k_p = 0$ to the dust-free case (perfect gas) and $M_A^{-2} = 0$ to a non-magnetic case. Also, 0.25 may be taken as a typical value of the ratio of specific heat of dust particles and specific heat at constant volume of the perfect gas (β').

The variation of the flow variables $\frac{u}{u_s}$, $\frac{\rho}{\rho_s}$, $\frac{p}{p_s}$ and $\frac{h}{h_s}$ for adiabatic case are shown in figures (1) to (4) and for isothermal case in figures (5) to (8). Table (1) shows the values of β and λ_p at various values of k_p , G_0 and M_A^{-2} . The density ratio β remains same in both the adiabatic and isothermal cases. The ratio of the velocity of the inner surface (piston) and the fluid velocity just behind the shock is $\frac{U_p}{u_s} = \frac{1}{(1-\beta)C} = \frac{\lambda_p}{(1-\beta)}$ which is always greater than 1 from table (1). Figure (2) shows that the reduced density $\frac{\rho}{\rho_s}$ at $M_A^{-2} = 0$ is rapidly decreased near the piston (inner contact surface) in the case of adiabatic flow; whereas this effect is removed in the case of isothermal flow (figure (6)).

Figure (9) shows that variation of λ_p with respect to k_p for different value of G_0 and M_A^{-2} . For $G_0 = 1$, λ_p noticeably decreases by an increase in k_p . It means that the strength of the shock is decreased when k_p is increased. For $G_0 = 100$, λ_p increases with increase in k_p . It means that the strength of the shock is increased by an increase in k_p . Physically it means that when $G_0 = 100$ the density of the perfect gas in mixture is highly decreased which overcomes the effect of incompressibility of the mixture and finally makes a small decrease in the distance between the piston and shock front, and an increase in the shock strength. Further when magnetic field is applied on flow-field, the value of λ_p is decreased which means that effect of magnetic field is to decrease the shock strength.

It is found that an increase in the value of k_p

- i. increases the density ratio across the shock $\beta (= \frac{\rho_0}{\rho_s})$ when $G_0 = 1$, but in case of $G_0 = 100$ the density ratio decreases (see table 1);
- ii. increases the distance of piston from the shock front when $G_0 = 1$, and decreases it when $G_0 = 100$ (see table 1).
- iii. increases the reduced fluid velocity $\frac{u}{u_s}$, the reduced density $\frac{\rho}{\rho_s}$ and the reduced pressure $\frac{p}{p_s}$ at any point in the flow-field behind the shock when $G_0 = 1$ and decreases these when $G_0 = 100$ (see figures 1, 2, 3 in adiabatic flow and 5, 6, 7 in isothermal flow); and
- iv. decreases the reduced magnetic field $\frac{h}{h_s}$ when $G_0 = 1$ and increases it when $G_0 = 100$ (see figure 4 in adiabatic flow and figure 5 in isothermal flow).

This shows that an increase in k_p decreases the shock strength when $G_0 = 1$ and increases it when $G_0 = 100$. Physical interpretations of these effects are as follows:

In the mixture, small solid particles of density equal to that of the perfect gas occupy a significant part of the volume which lowers the compressibility of the medium at $G_0 = 1$. Also, the compressibility of the mixture is reduced by an increase in k_p which causes an increase in the distance between the shock front and the piston, a decrease in the shock strength, and the above nature of the flow variables. In the mixture at $G_0 = 100$, small solid particles of density equal to 100 times that of the perfect gas occupy a very small portion of the volume, and therefore compressibility is not lowered much; the inertia of the medium is increased significantly due to dust load. An increase in k_p , from 0.1 to 0.4 in the mixture for $G_0 = 100$, means that the perfect gas constituting 90% of the total mass and occupying 99.889% of the total volume now constitutes 60% of the total mass and occupies 99.338% of the total volume. Due to this reason, the density of the perfect gas in mixture is highly decreased which overcomes the effect of incompressibility of the mixture and finally causes a small decrease in the distance between piston and shock front, an increase in the shock strength, and the above behavior of flow variables.

Effects of an increase in the value of G_0 are

- i. to decrease the value of β (i.e. to increase the shock strength)(see table 1);
- ii. to decrease the distance of piston from the shock front; and
- iii. to decrease the flow variables $\frac{u}{u_s}$, $\frac{\rho}{\rho_s}$ and $\frac{p}{p_s}$ and to increase $\frac{h}{h_s}$ (see figures 1, 2, 3, 4 in adiabatic flow and 5, 6, 7, 8 in isothermal flow).

These effects may be physically interpreted as follows:

Due to increase in G_0 (at constant k_p), there is high decrease in Z_0 , i.e. the volume fraction of solid particles in the mixture becomes comparatively very small. This effect induces comparatively more compression of the mixture in the region between shock and piston, which displays the above effect.

An increase in the value of the parameter for strength of the magnetic field M_A^{-2}

- i. decreases λ_p , i.e. increases the distance of the piston from the shock front. Physically it means that the gas behind the shock front is less compressed and the strength of the shock is decreased(see table 1);

- ii. increases the value of β (i.e. decreases the shock strength), which is same as given in (i) above (see table 1);
- iii. decreases the flow variables $\frac{u}{u_s}$ and $\frac{h}{h_s}$ at any point in the flow-field behind the shock front (see figures 1 and 4 (for adiabatic flow) and 5 and 8 (for isothermal flow)); and
- iv. increases the flow variable $\frac{\rho}{\rho_s}$ and $\frac{p}{p_s}$ (see figures 2 and 3 (for adiabatic flow) and 6 and 7 (for isothermal flow)).

Table 1. Variation of the density ratio $\beta(= \frac{\rho_0}{\rho_s})$ across the shock front and the position of the piston surface λ_p for different values of k_p , G_0 and M_A^{-2} with $\gamma = \frac{5}{3}$, $n = -0.15$.

β						
	$G_0 = 1$			$G_0 = 100$		
k_p	$M_A^{-2} = 0$	$M_A^{-2} = 0.005$	$M_A^{-2} = 0.01$	$M_A^{-2} = 0$	$M_A^{-2} = 0.005$	$M_A^{-2} = 0.01$
0.0	0.250000	0.255571	0.261039	0.250000	0.255571	0.261039
0.1	0.320408	0.323181	0.325991	0.245736	0.251445	0.257042
0.2	0.391045	0.392508	0.394002	0.240704	0.246579	0.252331
0.3	0.461983	0.462777	0.463587	0.234685	0.240763	0.246703
0.4	0.533333	0.533763	0.534201	0.227373	0.233704	0.239877
λ_p (Adiabatic Flow)						
0.0	0.902678	0.896105	0.896105	0.902676	0.896105	0.890519
0.1	0.875007	0.870632	0.866715	0.904397	0.897713	0.892047
0.2	0.845084	0.842048	0.839236	0.906424	0.899606	0.893846
0.3	0.812399	0.810235	0.808165	0.908845	0.901866	0.895992
0.4	0.776228	0.774649	0.773102	0.911787	0.904607	0.898592
λ_p (Isothermal Flow)						
0.0	0.912360	0.902459	0.895209	0.912360	0.902459	0.895209
0.1	0.884890	0.878382	0.873099	0.913830	0.903869	0.896585
0.2	0.854980	0.850634	0.846847	0.915560	0.905536	0.898203
0.3	0.822090	0.819150	0.816443	0.917650	0.907535	0.900148
0.4	0.785450	0.783453	0.781514	0.920190	0.909969	0.902518

Also, table 1 shows that the effect of magnetic field on shock strength, in both the cases (adiabatic and isothermal flows), decreases significantly on increasing the mass concentration of solid particles k_p at $G_0 = 1$; whereas at $G_0 = 100$ the effect of magnetic field on the shock strength is almost not influenced by increasing k_p . Thus the presence of magnetic field has decaying effect on the shock wave, but this effect is decreased on increasing k_p when $G_0 = 1$.

Comparison between Adiabatic and Isothermal Flows:

- i. In isothermal flow at $M_A^{-2} = 0$ (non-magnetic case) the density is almost constant in the flow-field behind the shock; whereas at $M_A^{-2} = 0.005$ and 0.01 (magnetic cases) the density decreases very rapidly near the piston (see figure 6). But in adiabatic flow in both the magnetic and non-magnetic cases ($M_A^{-2} = 0, 0.005, 0.01$) the density decreases very rapidly near the piston (see figure 2).
- ii. From table 1 it is clear that λ_p (position of the piston surface) in isothermal flow is greater than that in the adiabatic flow. Physically, it means that the gas is more compressed in the isothermal flow in comparison to that in adiabatic flow. Thus the strength of the shock is higher in the isothermal flow than that in the adiabatic flow.

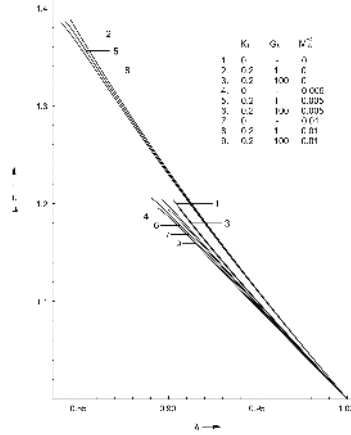


Figure 1: Variation of reduced velocity in the region behind the shock front for adiabatic flow.

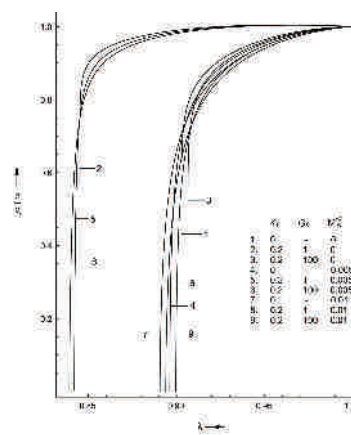


Figure 2: Variation of reduced density in the region behind the shock front for adiabatic flow.

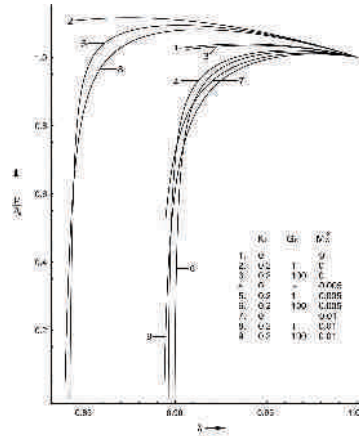


Figure 3: Variation of reduced pressure in the region behind the shock front for adiabatic flow.

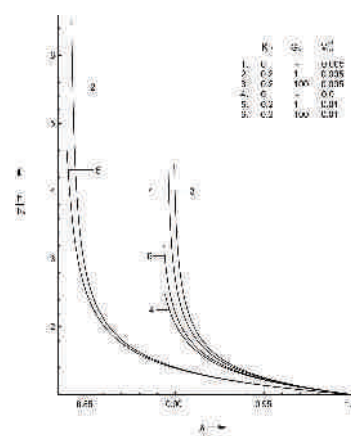


Figure 4: Variation of reduced magnetic field in the region behind the shock front for isothermal flow.

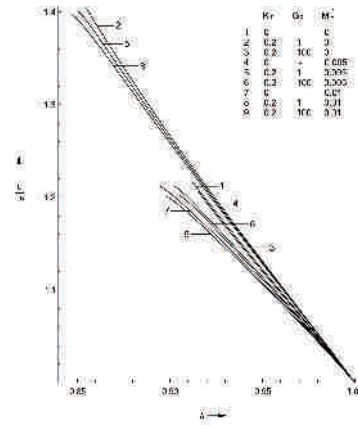


Figure 5: Variation of reduced velocity in the region behind the shock front for different time

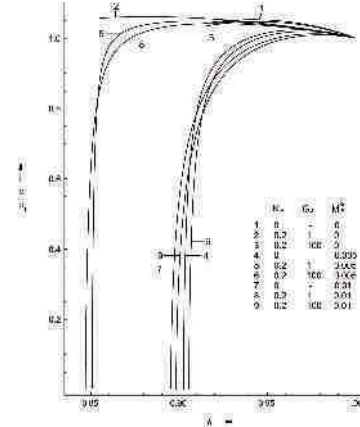


Figure 6: Variation of reduced density in the region behind the shock front for different time

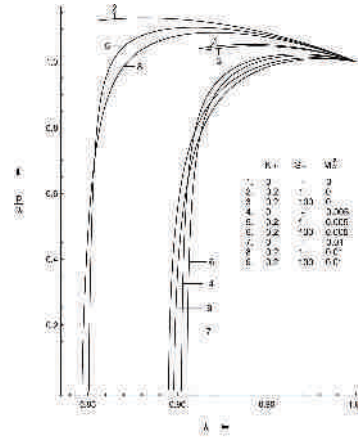


Figure 7: Variation of reduced pressure in the region behind the shock front for different time

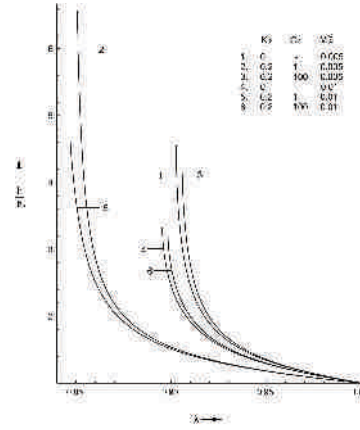


Figure 8: Variation of volume magnetic field in the region behind the shock front for different time

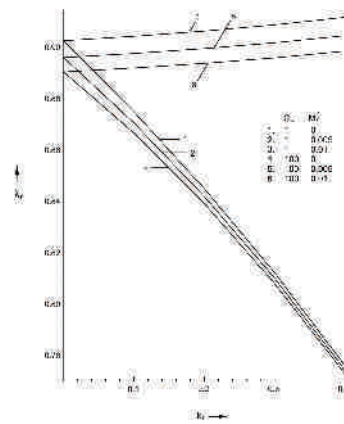


Figure 9: Variation of magnetic field with distance of solid particle in the region behind the shock front for different time

6. Conclusion

In this work, we have studied the self-similar solution for the flow behind a strong shock wave propagating in a perfectly conducting dusty gas in the presence of an azimuthal magnetic field. The shock is driven by a piston moving with velocity obeying a power law. On the basis of this work, one may draw the following conclusions:

- i. An increase in the mass concentration of solid particles (k_p) decreases the shock strength at lower values of G_0 , and increases it at its higher values. Also for $G_0 = 1$, it increases the reduced velocity, reduced density and reduced pressure and decreases the reduced magnetic field at any point in the flow-field behind shock; whereas for $G_0 = 100$, it decreases the reduced velocity, reduced density and reduced pressure and increases the reduced magnetic field.
- ii. An increase in the value of the ratio of the density of solid particles and the initial density of the perfect gas in the mixture (G_0) increases the shock strength and decreases the distance of piston from the shock front. Also, it decreases the reduced velocity, the reduced density and the reduced pressure and increases the reduced magnetic field at any point in the flow field behind the shock. These effects are more impressive at higher values of $k_p (= 0.4)$.
- iii. The presence of magnetic field decreases the reduced fluid velocity but increases the reduced pressure and reduced density at any point in the flow-field behind the shock. Also, the effect of magnetic field on shock strength, in both the cases (adiabatic and isothermal flow), decreases significantly by increasing k_p at $G_0 = 1$; whereas at $G_0 = 100$ the effect of magnetic field on the shock strength is almost not influenced by increasing k_p .
- iv. The value of λ_p (piston position) in isothermal flow is greater than that in the adiabatic flow i.e. the strength of the shock is higher in the isothermal flow than that in the adiabatic flow.

REFERENCES

- [1] Balick, B. and Frank, A. : Shapes and shaping of planetary nebulae, *Annu. Rev. Astron. Astrophys.*, 40 (2002), 439-486.
- [2] Bhutani, O. P. : Propagation and attenuation of a cylindrical blast wave in magnetogas-dynamics, *J. Math. Anal. Appl.*, 13 (1966), 565-576.
- [3] Christer, A. H. and Helliwell, J. B. : Cylindrical shock and detonation waves in magneto-gasdynamics, *J. Fluid Mech.*, 39 (1969), 705-725.

- [4] Cole, J. D. and Greifinger, C. : Smilarity solution for cylindrical magnetohydrodynamic blast waves, *Phys. Fluids* 5 (1962), 1557-1607.
- [5] Deb Ray, G. : Similarity solutions for cylindrical blast waves in magnetogasdynamics, *Phys. Fluids*, 16 (1973), 559-560.
- [6] Finkleman, D. and Baron, J. R. : A characteristic approach to radiative gas dynamics, *AIAA. J.* 8 (1970), 87-95.
- [7] Freeman, R. A. : Variable energy blast waves, *British J. Appl. Phys. (J. Phys. D.)*, 1 (1968), 1697-1710.
- [8] Freeman, R. A. and Craggs, J. D. : Shock waves from spark discharges, *British J. Appl. Phys. (J. Phys. D.)*, 1 (1969), 421-428.
- [9] Gretler, W. and Regenfelder, R. : Variable energy blast waves generated by a piston moving in a dusty gas, *J. Eng. Math.*, 52 (2005), 321-336.
- [10] Hartmann, L. : *Accretion Processes in Star Formation*, Cambridge Uni. Press, Cambridge (1998).
- [11] Helliwell, J. B. : Self-similar piston problem with radiative heat transfer, *J. Fluid Mech.*, 37 (1969), 497-512.
- [12] Higashino, F. and Suzuki, T. : The effect of particles on blast wave in a dusty gas, *Z. Naturforsch*, 35 a (1980), 1330-1336.
- [13] Korobeinikov, V. P. : *Problems in the theory of point explosion in gases*, American Mathematical Society, Providence (1976).
- [14] Laumbach, D. and Probstein, R. F. : Self similar strong shocks with radiation in a decreasing exponential atmosphere, *Phys. Fluids* 13 (1970), 1178-118.
- [15] Marble, F. E. : Dynamics of dusty gases, *Annu. Rev. Fluid Mech.* 2 (1970), 397-446.
- [16] Miura, R. and Glass, I. I. : Development of the flow induced by a piston moving impulsively in a dusty gas, *Proc. Roy. Soc. London A* 3 (1985), 295-309.
- [17] Nath, G., Vishwakarma, J. P., Srivastava, V. K. and Sinha, A. K. : Propagation of magnetogasdynamic shock waves in a self-gravitating gas with exponentially varying density, *J. Theo. Appl. Phys.* 7 (2013), 15.
- [18] Pai, S. I. : Cylindrical shock waves produced by instantaneous energy release in magnetogasdynamics, *Proc.Theor. Appl. Mech.*, (1958), 89-100.
- [19] Pai, S. I. : Two phase flow [chapter V], *Vieweg tracts in Pure Appl. Phys.*, Vol. 3 (1977), Braunenweig, ViewegVerlag.
- [20] Pai, S. I., Menon, S. and Fan, Z. Q. : Similarity solution of a strong shock wave propagation in a mixture of gas and dusty particles, *Int. J.Engng. Sci.* 18 (1980), 1365-1373.
- [21] Rosenau, P. and Frankenthal, S. : Equatorial propagation of axisymmetric magnetohydrodynamic shocks I, *Phys. Fluids*, 19 (1976), 1889-1899.
- [22] Sachdev, P. L. and Ashraf, S. : Converging spherical and cylindrical shocks with zero temperature gradient in the rear flow field, *ZAMP.*, 22 (1971), 1095-1102.
- [23] Sakurai, A. : Blast wave theory, *Basic developments in fluid dynamics* (Holt M. editor), vol. I (1965), New York : Academic Press, 309-375.
- [24] Sedov, L. I. : *Similarity and dimensional methods in mechanics*, Academic Press, New York, (1985).
- [25] Singh, L. P., Sharma, V. D. and Ram, R. : Flow pattern induced by a piston moving in a perfectly conducting inviscid radiating gas, *Phys. Fluids B.*, 1 (1989), 692-699.

- [26] Steiner, H. and Hirschler, T. : A self-similar solution of a shock propagation in a dusty gas, *Eur. J. Mech. B. Fluids*, 21 (2002), 371-380.
- [27] Taylor, G. I. : The formation of a blast wave by a very intense explosion, *Proc. R. Soc., London*, 201 (1950), 159-174.
- [28] Vishwakarma, J. P. : Propagation of shock waves in a dusty gas with exponentially varying density, *Eur. Phys. J. B.*, 16 (2000), 369-372.
- [29] Vishwakarma, J. P., Nath, G. and Srivastava, R. K. : Self-similar solution for cylindrical shock waves in a weakly conducting dusty gas, *Ain. Shams. Eng. J.*, <http://dx.doi.org/10.1016/j.asej.2016.09.013> (2017).
- [30] Vishwakarma, J. P. and Pandey, S. N. : Propagation of strong spherical shock waves in a dusty gas, *Phys. Scr.*, 68 (2003), 259-263.
- [31] Vishwakarma, J. P. and Patel, N. : Magenotogasdynamical cylindrical shock waves in a rotating non-ideal gas with radiation heat flux, *J. Eng. Phys. Thermophys.*, 88 (2015), 521-530.
- [32] Vishwakarma, J. P. and Singh, A. K. : Self-similar solutions for cylindrical shock waves in a low conducting gas, *Aligarh Bull. Maths.* 28 (2009), 5-14.
- [33] Vishwakarma, J. P. and Yadav, A. K. : Self-similar analytical solution for blast waves in inhomogeneous atmospheres with frozen-in-magnetic field, *Eur. Phys. J. B.*, 34 (2003), 247-253.
- [34] Wang, K. C. : Approximate solution of a plane radiating piston problem, *J. Fluid Mech.*, 20 (1964), 447-455.
- [35] Whitham, G. B. : On the propagation of shock waves through region of non-uniform area or flow, *J. Fluid Mech.*, 4 (1958), 337-360.