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On Generalized $\phi-$recurrent Lorentzian $\beta-$Kenmotsu Manifolds

Kripasindhu Prasad

Department of Mathematics
Thakur Ram Multiple Campus, Birgunj Tribhuvan University, Nepal
e-mail:kripasindhuchaudhary@gmail.com
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Abstract

The present study deals with generalised $\phi-$recurrent Lorentzian $\beta-$Kenmotsu manifolds. We also find the necessary and sufficient condition for locally generalised $\phi-$recurrent lorentzian $\beta-$Kenmotsu manifolds.

1. Introduction

In (1991) De, U. C. and Guha N. [4] introduced the notion of generalised recurrent manifolds. Further De, U.C., Shaikh, A. A. and Biswas, S. [3] introduced the notion of phi recurrent sasakian manifolds. generalized $\phi-$recurrent (k, μ) contact metric manifold were studied by Jun, J B., Yildiz, A. and De, U. C. [7]. Also Bagewadi, C. S. and Prakasha, D. G. [5] studied generalised $\phi-$recurrent sasakian manifolds. Recently Prakasha, D. G. and Yildiz, A. [8] stuied the generalied $\phi-$recurrent lorentzian $\alpha-$Sasakian manifolds. Motivated from, in this paper we have study generalised $\phi-$recurrent lorentzian $\beta-$Kemmotsu manifold and obtain some interesting results.

This paper is organized as follows: in section 2, we have given a brief account of Lorentzian $\beta-$Kenmotsu manifolds. In section 4, we show that a generalized $\phi-$recurrent lorentzian $\beta-$Kenmotsu manifolds is an Einstein manifold. Further in the same section we also shown that in a generalised $\phi-$recurrent lorentzian $\beta-$ Kenmotsu manifolds the characteristic vector ξ and the associated vector field $\beta^2\rho_1+\rho_2$ are codirectional. In the same section, we also find the necessary and sufficient condition for locally generalised $\phi-$recurrent lorentzian $\beta-$Kenmotsu manifolds. In the same section, we study 3-dimensional generalised $\phi-$recurrent lorentzian $\beta-$Kenmotsu manifolds and shown is of constant curvature.

2. Preliminaries

An $(2n+1)$ dimensional differentiable manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is called a Lorentzian $\beta-$Kenmotsu manifolds with the structure (ϕ, ξ, η, g) where $\beta-$ is a smooth function on M if it admits a tensor field $\phi\eta$ of type $(1,1)$, a contravariant vector field ξ, η 1-form and a Lorentzian metric g which satisfy [1, 2, 3]

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$\text{(a)} \quad \eta(\xi) = -1, \quad \text{(b)} \quad g(X, \xi) = \eta(X), \quad \text{(c)} \quad \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$(D_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X, \tag{2.4}$$

$$D_X \xi = \beta(x - \eta(X)\xi), \tag{2.5}$$

$$(D_X \eta)(Y) = \beta[g(X, Y) - \eta(X)\eta(Y)], \tag{2.6}$$

where D denotes the operator of covariant differentiation with respect to g.

Also in lorentzian $\beta-$Kenmotsu manifolds, the following holds [1, 2, 3]

$$\eta(R(X, Y)Z) = g(R(X, Y)Z, \xi) = \beta^2[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)], \tag{2.7}$$

$$R(X, Y)\xi = \beta^2[\eta(X)Y - \eta(Y)X], \tag{2.8}$$

$$S(\phi X, \phi Y) = S(X, -2n\beta^2\eta(X)\eta(Y), Y), \tag{2.9}$$

$$S(X, \xi) = -2n\beta^2\eta(X) \tag{2.10}$$

for all vector fields X, Y, Z, where S is the Ricci tensor of type $(0, 2)$ and R is the Riemannian curvature tensor of the manifold.

Definition (2.1). A Lorentzian $\beta-$Kenmotsu manifolds is said to be a locally $\phi-$symmetric manifold if [8]

$$\phi^2((D_W R)(X, Y)Z) = 0, \tag{2.11}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Definition (2.2). A lorentzian $\beta-$Kenmotsu manifolds is said to be a $\phi-$ recurrent manifold if there exists a nowhere vanishing 1 form A such that [4, 5]

$$\phi^2((D_W R)(X, Y)Z) = A(W)R(X, Y, Z), \tag{2.12}$$

for all vector fields $X, Y, Z, W \in T(M)$.

Definition (2.3). A Lorentzian $\beta-$Kenmotsu manifolds is said to be a generalised $\phi-$recurrent manifold if its curvature tensor R satisfies [4, 5]

$$\phi^2((D_W R)(X, Y)Z) = A(W)R(X, Y, Z) + B(W)[g(Y, Z)X - g(X, Z)Y], \tag{2.12}$$

where A and B are 1 forms, B is non zero and these are defined as $A(X) = g(X, \rho_1)$ and $B(X) = g(X, \rho_2)$.

3. Generalised $\phi-$recurrent lorentzian $\beta-$Kenmotsu manifolds

Let us consider a generalised $\phi-$recurrent lorentzian $\beta-$Kenmotsu manifolds. Then by virtue of (2.1) and (2.12), we get

$$\begin{aligned}(D_W R)(X,Y)Z &= \eta((D_W R)(X,Y)Z)\xi \\ &= A(W)R(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y]\end{aligned} \tag{3.1}$$

from which it follows that

$$\begin{aligned}&g((D_W R)(X,Y)Z,U) + \eta((D_W R)(X,Y)Z)\eta(U) \\ &= A(W)g(R(X,Y)Z,U) + B(W)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)].\end{aligned} \tag{3.2}$$

Let $\{e_i\}$, $i = 1, 2, \dots, 2n+1$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (3.2) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$\begin{aligned}(D_W S)(Y,Z) = &-\sum_{n=1}^{2n+1} \eta((D_W R)(e_i,Y)Z)\eta(e_i,\xi) + A(W)S(Y,Z) \\ &+ 2nB(W)g(Y,Z).\end{aligned} \tag{3.3}$$

Replacing Z by ξ in (3.3) and using (2.5) and (2.10), we get

$$(D_W S)(Y,\xi) = -\sum_{n=1}^{2n+1} \eta((D_W R)(e_i,Y)\xi)\eta(e_i,\xi) + A(W)S(Y,\xi) + 2nB(W)g(Y,\xi). \tag{3.4}$$

After simplification the first term

$$\sum_{n=1}^{2n+1} \eta((D_W R)(e_i,Y)\xi)\eta(e_i,\xi) = 0. \tag{3.5}$$

Then using (3.5) in (3.4) we have

$$(D_W S)(Y,\xi) = 2n[-\beta^2 A(W) + B(W)]\eta(Y). \tag{3.6}$$

Now we have Using (2.5) and (2.10) in the above relation, it follows that

$$(D_W S)(Y,\xi) = -2n\beta^2 g(Y,W) - \beta S(Y,W). \tag{3.7}$$

In view of (3.6) and (3.7) we have

$$S(Y,W) = -2n\beta^2 g(Y,W) + 2n[-\beta^2 A(W) - B(W)]\eta(Y). \tag{3.7}$$

Replacing Y by ϕY and W by ϕW in above equation and using (2.3) and (2.9), we get

$$S(Y,W) = -2n\beta^2 g(Y,W),$$

Hence, we can state the following theorem:

Theorem (3.1). A generalized $\phi-$recurrent lorentzian $\beta-$Kenmotsu manifold (M^{2n+1}, g) is an Einstein manifold.

Now from (2.1), we have

$$\begin{aligned}(D_W R)(X,Y)Z = &- \eta((D_W R)(X,Y)Z)\xi + A(W)R(X,Y)Z \\ &+ B(W)[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{3.8}$$

From (2.7) and using the Bianchi's second identity, we get

$$\begin{aligned}&A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z) \\ &+ B(W)[g(Y,Z)X - g(X,Z)Y] + B(X)[g(W,Z)Y - g(Y,Z)W] \\ &+ B(Y)[g(X,Z)W - g(W,Z)X] = 0.\end{aligned} \tag{3.9}$$

Now using (2.9) we get from (3.9)

$$\begin{aligned}&\{-\beta^2 A(W) + B(W)\}[g(Y,Z)X - g(X,Z)Y] \\ &+ \{-\beta^2 A(X) + B(X)\}[g(W,Z)Y - g(Y,Z)W] \\ &+ \{-\beta^2 A(W) + B(W)\}[g(X,Z)W - g(W,Z)X] = 0.\end{aligned} \tag{3.10}$$

Putting $Y = Z = e_i$ in (3.10) and taking summation over i, $1 \le i \le 2n+1$, and simplifying, we get

$$\{-\beta^2 A(W) + B(W)\}\eta(X) = \{-\beta^2 A(X) + B(X)\}\eta(W), \tag{3.11}$$

for all vector fields X, W.

Replacing X by ξ in (3.10), we get

$$\{-\beta^2 A(W) + B(W)\} = -\eta(W)\{-\beta^2\eta(\rho_1) + \eta(\rho_2)\}, \tag{3.12}$$

for any vector field W, where $A(\xi) = g(\xi,\rho_1) = \eta(\rho_1)$ and $B(\xi) = g(\xi,\rho_2) = \eta(\rho_2)$, ρ_1, ρ_2 being the vector field associated to the 1-form A and B i.e.,$A(X) = g(X,\rho_1)$ and $B(X) = g(X,\rho_2)$.

From (3.11) and (3.12), we can state the following theorem:

Theorem (3.2). In a generalised $\phi-$recurrent lorentzian $\beta-$Kenmotsu manifold (M^{2n+1}, g) $n \ge 1$, the characteristic vector field ξ and the vector field $-\beta^2\rho_1 + \rho_2$

associated to the 1-form $-\beta^2 A + B$ are co-directional and the 1-form A is given by $\{-\beta^2 A(W) + B(W)\} = -\eta(W)\{-\beta^2\eta(\rho_1) + \eta(\rho_2)\}$.

Theorem (3.3). A lorentzian $\beta-$Kenmotsu manifold (M^{2n+1}, g) $n \geq 1$ is locally generalised $\phi-$recurrent if and only if the relation

$$\begin{aligned}(D_W R)(X,Y)Z =&\beta\{\beta^2[g(X,W)g(Y,Z) - g(Y,W)g(X,Z)]\xi \\ &- g(R(X,Y)W,Z)\xi\} + A(W)R(X,Y)Z \\ &+ B(W)\{g(Y,Z)X - g(X,Z)Y\}\end{aligned} \tag{3.13}$$

holds for all horizontal vector fields X, Y, Z, W on manifold.

Proof: In view of (2.5) and (2.6) it can be easily seen that in a lorentzian $\beta-$Kenmotsu manifold the following relation holds

$$(D_W R)(X,Y)\xi = \beta\{\beta^2[g(X,W)Y - g(Y,W)X] - R(X,Y)W\}. \tag{3.14}$$

Now, using the properties of curvature tensor, we have [9]

$$\begin{aligned}g((D_W R)(X,Y)Z,U) =&g(D_W R(X,Y)Z,U) + R(D_W X,Y,U,Z) \\ &+ R(X,D_W Y,U,Z) + R(X,Y,Z,D_W Z)\end{aligned} \tag{3.15}$$

where $(X,Y,Z,U) = g(R(X,Y)Z,U)$. Since D is metric connection then we have

$$g(D_W R(X,Y)Z,U) = g(R(X,Y)D_W U,Z) - D_W g(R(X,Y)U,Z), \tag{3.16}$$

$$D_W g(R(X,Y)U,Z) = g(D_W R(X,Y)U,Z) + g(R(X,Y)U,D_W Z). \tag{3.17}$$

Using (3.16) and (3.17), we get

$$\begin{aligned}g(D_W R(X,Y)Z,U) =&- g(D_W R(X,Y)U,Z) - g(R(X,Y)U,D_W Z) \\ &+ g(R(X,Y)D_W U,Z).\end{aligned} \tag{3.18}$$

Using (3.18) in (3.15), we get

$$g((D_W R)(X,Y)Z,U) = -g((D_W R)(X,Y)U,Z). \tag{3.19}$$

Using (3.19) and in view of (2.1) and (2.12), we have

$$\begin{aligned}(D_W R)(X,Y)Z =&g((D_W R)(X,Y)\xi,Z)\xi + A(W)R(X,Y,Z) \\ &+ B(W)[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{3.20}$$

Using (3.14) in (3.20) we get relation (3.13). Conversely if in a lorentzian $\beta-$Kenmotsu manifold the relation (3.13) holds then apply ϕ on both side of (3.14) and taking Y, Z, X and W orthogonal to ξ, we get (2.12). This complete the proof of the theorem.

Theorem (3.4). A lorentzian $\beta-$Kenmotsu manifold is of constant curvature if and only if the relation

$$\phi^2((D_W R)(X,Y)\xi) = A(W)R(X,Y)\xi + B(W)[\eta(Y)X - \eta(X)Y] \tag{3.21}$$

hold for all horizontal vector fields X, Y, W.

Proof: In View of (2.1), (3.21) can be written as

$$(D_W R)(X,Y)\xi + \eta((D_W R)(X,Y)\xi)\xi = A(W)R(X,Y)\xi + B(W)[\eta(Y)X - \eta(X)Y]. \tag{3.22}$$

Now using (2.8), (3.13) in (3.22), we get

$$(D_W R)(X,Y)\xi = 0 \tag{3.23}$$

for any horizontal vector fields X, Y, W.

In view of (3.14) and (3.23), we have

$$R(X,Y)W = \beta^2[g(X,W)Y - g(Y,W)X] \tag{3.24}$$

for any vector fields X, Y,W orthogonal to ξ.

Conversely, if a lorentzian $\beta-$Kenmotsu manifold is of constant curvature, then from (3.24) the relation (3.21) holds. This proves the theorem.

4. 3-dimensional locally generalised $\phi-$recurrent lorentzian $\beta-$Kenmotsu manifolds

In a 3-dimensional lorentzian $\beta-$Kenmotsu manifold the curvature tensor is given by

$$\begin{aligned} R(X,Y)Z =& g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &+ \frac{r}{2}[g(X,Z)Y - g(Y,Z)X]. \end{aligned} \tag{4.1}$$

Now putting $Z = \xi$ in (4.1) and using (2.2), (2.12) the curvature tensor a 3-dimensional lorentzian $\beta-$Kenmotsu manifold is given by

$$\begin{aligned} R(X,Y)Z = & \left(\frac{r}{2} + 2\beta^2\right)[g(Y,Z)X - g(X,Z)Y] + \left(\frac{r}{2} + 3\beta^2\right)[g(Y,Z) \\ & \eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{4.2}$$

Taking the covariant derivative of both side of (4.2), we get

$$
\begin{aligned}
(D_W R)(X,Y)Z = & \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y - g(Y,Z)\eta(X)\xi \\
& + g(X,Z)\eta(Y)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y] \\
& + \left(\frac{r}{2} + 3\beta^2\right)[g(Y,Z)(D_W\eta)(X)\xi + g(Y,Z)\eta(X) \\
& (D_W\xi) - g(X,Z)(D_W\eta)(Y)\xi - g(X,Z)\eta(Y)(D_W\xi) \\
& + (D_W\eta)(Y)\eta(Z)X + (D_W\eta)(Z)\eta(Y)X \\
& - (D_W\eta)(X)\eta(Z)X - (D_W\eta)(Z)\eta(X)Y].
\end{aligned} \tag{4.3}
$$

Now taking all vector fields X, Y, Z, W are orthogonal to ξ and using (2.5) and (2.6), we get

$$
\begin{aligned}
(D_W R)(X,Y)Z = & \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] + \left(\frac{r}{2} + 3\beta^2\right)\beta \\
& [g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\xi.
\end{aligned} \tag{4.4}
$$

From (4.4), it follows that

$$
\phi^2(D_W R)(X,Y)Z = \frac{dr(W)}{2}[g(Y,Z)\phi^2 X - g(X,Z)\phi^2 Y].
$$

Since, we have taken X, Y, Z as horizontal vector field and using (2.1), (2.5) and (2.6), we get

$$
\phi^2(D_W R)(X,Y)Z = \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y]. \tag{4.5}
$$

Then in view of (2.12), equation (4.5) reduces to

$$
A(W)R(X,Y)Z = \left[\frac{dr(W)}{2} - B(W)\right][g(Y,Z)X - g(X,Z)Y]. \tag{4.6}
$$

Putting $W = e_i$ where $\{e_i\}$, $i = 1, 2, 3$ is an orthonormal basis of the tangent space at any point of the manifold and taking summation over i, we get

$$
R(X,Y)Z = \lambda[g(Y,Z)X - g(X,Z)Y]
$$

where $\lambda = \left[\frac{dr(e_i)}{2A(e_i)} + \beta^2\right]$ is a scalar, since A is a non zero 1-form. Then by Schur's theorem λ will be a constant of the manifold. Therefore we state the following theorem:

Theorem (4.1). A 3-dimensional locally generalied $\phi-$recurrent Lorentzian $\beta-$Kenmotsu manifolds is a manifold of constant curvature.

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