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On A Semi-Symmetric Non Metric Connection in Lorentzian Para-Cosympletic Manifold

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Abstract

In this paper we define and studied a semi-symmetric non metric connection on a Lorentzian Para-Cosympletic Manifold and prove its existence. We deduce the expression for curvature tensor and Ricci tensor of semi-symmetric non metric connection defined. A necessary and sufficient condition has been deduced for the Ricci tensor to be symmetric and skew-symmetric under certain condition. Bianchi first identity associated with the connection, Einstein Manifold, Weyl conformal curvature tensor of the same connection were found.

Keywords and Phrases : Semi-symmetric non-metric connection, Ricci tensor, conformal curvature tensor, cosymplectic manifold.
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1. Introduction

Let (M^n, g) be a $n-$dimensional differentiable manifold on which there are defined a tensor field ϕ of type (1,1) a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g which satisfy

$$\phi^2 X = X + \eta(X)\,\xi \tag{1.1}$$

$$\eta(\xi) = -1 \tag{1.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \tag{1.3}$$

$$g(X, \xi) = \eta(X) \tag{1.4}$$

Then M^n is called a Lorentzian Para-contact Manifold (or LP-contact Manifold) and the structure(ϕ, ξ, η, g) is called an LP-contact structure (Matsumoto 1989).

In an LP-Contact Manifold, we have

(1.5)(a) $\phi\xi = 0$

(b) $\eta(\phi X) = 0$

(c) rank $\phi = n-1$.

Let us put

(1.6) $$F(X,Y) = g(\phi X, Y)$$

Then the tensor field F is symmetric (0,2) tensor field

(1.7) $$F(X,Y) = F(Y,X)$$

An LP-contact manifold is said to be an LP-cosympletic manifold (Prasad & Ojha 1994) if

(1.8) $$D_X\phi = 0 \Rightarrow D_X F(Y,Z) = 0$$

On this manifold, we have

(1.9) $$(D_X\eta)(Y) = 0$$

and

(1.10) $$D_X\xi = 0$$

For vector field X, Y and Z where D_X denotes covariant differentiation with respect to g.

2. Semi- Symmetric Non Metric Connection in an LP-Cosympletic Manifold

Let (M^n, g) be an LP-cosympletic manifold with Levi-Civita connection D. We define a linear connection $\overline{D}$ on M^n by

(2.1) $$\overline{D}_X Y = D_X Y + \eta(Y)X + a(X)Y$$

where η and a are 1-form associated with vector field ξ and A on M^n given by

(2.2) $$g(X,\xi) = \eta(X)$$

and

(2.3) $$g(X,A) = a(X)$$

for all vector field $X \,\epsilon \chi(M^n)$ where $\chi(M^n)$ is the set of all differentiable vector field on M^n.

Using (2.1) the torsion tensor $\overline{T}$ of M^n with respect to the connection $\overline{D}$ is given by

$$\overline{T}(X,Y) = \eta(Y)X - \eta(X)Y + a(X)Y - a(Y)X \tag{2.4}$$

A linear connection satisfying (2.4) is called a semi-symmetric connection. Further using (2.1) we have

$$(\overline{D}_X g)(Y,Z) = -\eta(Y)g(X,Z) - \eta(Z)g(X,Y) - 2a(X)g(Y,Z). \tag{2.5}$$

A linear connection $\overline{D}$ defined by (2.1) and satisfying (2.4) and (2.5) is called a semi-symmetric non metric connection.

Let $\overline{D}$ be a linear connection in M^n given by

$$\overline{D}_X Y = D_X Y + H(X,Y). \tag{2.6}$$

Now we shall determine the tensor field H such that $\overline{D}$ satisfies (2.4) and (2.5)

From (2.6), we have

$$\overline{T}(X,Y) = H(X,Y) - H(Y,X), \tag{2.7}$$

Denote

$$G(X,Y,Z) = (\overline{D}_X g)(Y,Z). \tag{2.8}$$

From (2.6) and (2.8), we have

$$g(H(X,Y),Z) + g(H(X,Z),Y) = -G(X,Y,Z). \tag{2.9}$$

From (2.6), (2.8), (2.9) and (2.5) we have

$$g(\overline{T}(X,Y),Z) + g(\overline{T}(Z,X),Y) + g(\overline{T}(Z,Y),X) = g(H(X,Y),Z)$$

$$-g(H(Y,X),Z)+g(H(Z,X),Y-g(H(X,Z),Y)+g(H(Z,Y),X)-g(H(Y,Z),X)$$

$$= 2g(H(X,Y),Z) + G(X,Y,Z) + G(Y,X,Z) - G(Z,X,Y)$$

$$= 2g(H(X,Y),Z)-2\eta(Z)g(X,Y)-2a(X)g(Y,Z)-2a(Y)g(X,Z)+2a(Z)g(X,Y)$$

Or

$$H(X,Y) = \frac{1}{2}\{'\overline{T}(X,Y) + '\overline{T}(X,Y) + '\overline{T}(Y,X)\} + a(X)Y + a(Y)X$$

$$+g(X,Y)\xi - g(X,Y)A$$

Where $'\overline{T}$ be a tensor field of type (1, 2) defined by

$$g('\overline{T}(X,Y),Z) = g(\overline{T}(Z,X),Y)$$

Or

$$H(X,Y) = \eta(Y)X + a(X)Y$$

This implies

$$\overline{D}_X Y = D_X Y + \eta(Y)\, X + a(X)\, Y.$$

Thus we have the following theorem :

Theorem (2.1) : Let (M^n, g) be an LP-cosympletic manifold with almost Lorentzian para contact metric structure (ϕ, ξ, η, g) admitting a semi-symmetric non metric connection $\overline{D}$ which satisfies (2.4) and (2.5) then the semi-symmetric non metric connection is given by

$$\overline{D}_X\, Y = D_X\, Y + \eta(Y)\, X + a(X)Y.$$

3. Existence of semi-symmetric non metric connection $\overline{D}$ in an LP-cosympletic manifold

Let X, Y, Z be any three vector fields on an $LP-$cosympletic manifold (M^n, g) with almost Lorentzian para contact metric structure (ϕ, ξ, η, g). We define a connection $\overline{D}$ by the following equation :

(3.1)

$$\begin{aligned} 2g(\overline{D}_X Y, Z) = {} & Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z) - g([Y,Z],X) \\ & + g([Z,X],Y) + g(\eta(Y)X - \eta(X)Y + a(X)Y - a(Y)X, Z) \\ & + g(\eta(X)z - \eta(Z)X + a(Z)X - a(X)Z, Y) \\ & + g(\eta(Y)Z - \eta(Z)Y + a(Y)Z - a(Z)Y, X) \end{aligned}$$

Which holds for all vector fields $X, Y, Z \in \chi(M^n)$.

It can easily be verified that the mapping

$$\overline{D} \;:\; (X,Y) \to \overline{D}_X\, Y$$

satisfying the following identities

$$\overline{D}_X(Y+Z) = \overline{D}_X Y + \overline{D}_X Z \tag{3.2}$$

$$\overline{D}_{X+Y} Z = \overline{D}_X Z + \overline{D}_Y Z \tag{3.3}$$

$$\overline{D}_{fX} Y = f\overline{D}_X Y \tag{3.4}$$

$$\overline{D}_X fY = f\overline{D}_X + (Xf)Y \tag{3.5}$$

for all $X, Y, Z \in \chi(M^n)$ and for all $f \in F(M^n)$, the set of all differentiable mapping over M^n. From (3.2), (3.3), (3.4) and (3.5) we conclude that $\overline{D}$ determines a linear connection on M^n. Now from (3.1) we have

$$\overline{D}_X Y - \overline{D}_Y X - [X,Y] = \eta(Y)\,X - \eta(X)\,Y + a(X)\,Y - a(Y)\,X \tag{3.6}$$

Or

$$\overline{T}(X,Y) = \eta(Y)\,X - \eta(X)\,Y + a(X)\,Y - a(Y)\,X$$

Also, we have from (3.1)

$$\begin{aligned} 2g(\overline{D}_X Y, Z) + 2g(\overline{D}_X Z, Y) =& 2Xg(Y,Z) + 2\eta(Y)g(X,Z) \\ &+ 2\eta(Z)g(X,Y) + 4a(X)g(Y,Z) \end{aligned}$$

i.e.

$$(\overline{D}_X g)(Y,Z) = -\eta(Y)g(X,Z) - 2\eta(Z)g(X,Y) - 2a(X)g(Y,Z) \tag{3.7}$$

From (3.6) and (3.7) it follows that $\overline{D}$ determines a semi-symmetric non metric connection on (M^n, g). it can be easily verified that $\overline{D}$ determines a unique semi-symmetric non metric connection on (M^n, g).

Thus we have

Theorem (3.1) : Let (M^n, g) be an $LP-$Cosympletic manifold with an almost Lorentzian para-contact metric structure (ϕ, ξ, η, g) on it. Then there exist a unique linear connection $\overline{D}$ satisfying (2.4) and (2.5).

The above theorem proves the existence of a semi-symmetric non metric connection in an LP cosympletic manifold.

4. Curvature tensor of an $LP-$Cosympletic manifold with respect to the semi symmetric non metric connection $\overline{D}$

Let $\overline{R}$ and R be the curvature tensor of the connections $\overline{D}$ and D respectively then

$$\overline{R}(X,Y)Z = \overline{D}_X \overline{D}_Y Z - \overline{D}_Y \overline{D}_X Z - \overline{D}_{[X,Y]} Z. \tag{4.1}$$

From (2.1) and (4.1) we get

$$\begin{aligned} \overline{R}(X,Y,Z) =& \overline{D}_X (D_Y Z + \eta(Z)Y + a(Y)Z) - \overline{D}_Y (D_X Z + \eta(Z)X - a(X)Z) \\ &- D_{[X,Y]} Z - \eta(Z)[X,Y] - a([X,Y])Z. \end{aligned} \tag{4.2}$$

Using (1.9) in (4.2), we get

$$\overline{R}(X,Y,Z) = R(X,Y,Z) + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + da(X,Y)Z \tag{4.3}$$

where

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z$$

is the curvature tensor of D with respect to Riemannian connection. Contracting (4.3) we find

$$\overline{S}(Y,Z) = S(Y,Z) + \eta(Y)\eta(Z)n - \eta(Y)\eta(Z) + da(Z,Y) \tag{4.4}$$

Contracting with respect to Z we get

$$\overline{Q}_Y = Q_Y + n\eta(Y)\xi - \eta(Y)\xi - da(Y).$$

Again contracting w.r.t. Y

$$\overline{r} = (r+1) - n + \lambda. \tag{4.5}$$

Theorem (4.1) : The curvature tensor $\overline{R}(X,Y)Z$, the Ricci tensor $\overline{S}(Y,Z)$ and the scalar curvature $\overline{r}$ of an LP-Cosympletic manifold with respect to the semi-symmetric non metric connection $\overline{D}$ is given by(4.3),(4.4) and (4.5) respectively.

Let us assume that $\overline{R}(X,Y)Z = 0$ in (4.3) and contracting w.r.t. X we get

$$S(Y,Z) = \eta(Y)\eta(Z) - \eta(Y0\eta(Z)n - da(Z,Y).$$

Which again on contracting gives

$$r = 1 + n - \lambda \tag{4.6}$$

Hence we have

Theorem (4.2) : If the curvature tensor of an LP-Cosympletic manifold M^n admitting semi-symmetric non metric connection vanishes, then its scalar curvature is given by(4.6).

5. Symmetric and skew-symmetric condition of Ricci tensor of $\overline{D}$ in an LP-Cosympletic manifold

From (4.4) we have

$$\overline{S}(Z,Y) = S(Z,Y) + \eta(Y)\eta(Z)n - \eta(Y)\eta(Z) + da(Y,Z). \tag{5.1}$$

From (4.4) and (5.1) we have

$$\overline{S}(Y,Z) - \overline{S}(Z,Y) = da(Y,Z) - da(Z,Y). \tag{5.2}$$

If $\overline{S}(Y,Z)$ is symmetric, then the L.H.S. of (5.2) vanishes and we have

$$da(Y,Z) = da(Z,Y). \tag{5.3}$$

More over, if the relation (5.3) holds, then from (5.2) $\overline{S}(Y,Z)$ is symmetric. Hence we have

Theorem (5.1) : The Ricci tensor $\overline{S}(Y,Z)$ of the manifold with respect to the semi-symmetric non metric connection in an LP-cosympletic manifold is symmetric if and only if the relation (5.3) holds.

Again from (4.4) and (5.1), we find

(5.4)
$$\overline{S}(Y,Z)+\overline{S}(Z,Y) = 2S(Y,Z)+2\eta(Y)\eta(Z)n-2\eta(Y)\eta(Z)+da(Z,Y)+da(Y,Z).$$

If $\overline{S}(Y,Z)$ is skew-symmetric then the L.H.S. of (5.4) vanishes and we get

$$S(Y,Z) = \eta(Y)\eta(Z) - n\,\eta(Y)\eta(Z) - \frac{1}{2}da(Z,Y) - \frac{1}{2}da(Y,Z). \tag{5.5}$$

More over, if $S(Y,Z)$ is given by (5.5) then from (5.4) we get

$$\overline{S}(Y,Z) + \overline{S}(Z,Y) = 0$$

i.e. the Ricci tensor of $\overline{D}$ is skew-symmetric. Hence, we have

Theorem (5.2) : If an LP-cosympletic manifold admits a semi-symmetric non-metric connection $\overline{D}$ then a necessary and sufficient condition for the Ricci tensor of $\overline{D}$ to be skew-symmetric, that is the Ricci tensor of the Levi-civita connection D is given by (5.5).

6. Bianchi first identity associated with semi-symmetric non-metric connection $\overline{D}$ in an LP-cosympletic manifold

From (2.4), we have

$$\overline{T}(X,Y,Z) + \overline{T}(Y,Z,X) + \overline{T}(Z,X,Y) = 0, \tag{6.1}$$

where

$$\overline{T}(X,Y,Z) = g(\overline{T}(X,Y),Z).$$

Again from (2.4) we have

$$\begin{aligned} &\overline{T}(\overline{T}(X,Y),Z) + \overline{T}(\overline{T}(Y,Z),X) + \overline{T}(\overline{T}(Z,X),Y) \\ &= \eta(Y)a(X)Z - \eta(X)a(Y)Z + a(X)a(Y)Z - a(Y)a(X)Z \\ &+ \eta(Z)a(Y)X - \eta(Y)a(Z)X + a(Y)a(Z)X - a(Z)a(Y)X \\ &+ \eta(X)a(Z)Y - \eta(Z)a(X)Y + a(Z)a(X)Y - a(X)a(Z)Y \end{aligned} \tag{6.2}$$

and

$$(6.3)\quad (\overline{D}_X\overline{T})(Y,Z)+(\overline{D}_Y\overline{T})(Z,X)+(\overline{D}_Z\overline{T})(X,Y)$$
$$= da(X,Y)Z+da(Y,Z)X+da(Z,X)Y+\ a(Z)\eta(Y)X-a(Y)\eta(Z)X$$
$$-\,a(X)\eta(Y)Z-a(X)a(Y)Z+a(X)\eta(Z)Y+a(X)a(Z)Y+\ a(X)\eta(Z)Y$$
$$-\,a(Z)\eta(X)Y-a(Y)\eta(Z)X-a(Y)a(Z)X+a(Y)\eta(X)Z+a(Y)a(X)Z$$
$$+\ a(Y)\eta(X)Z-a(X)\eta(Y)Z-a(Z)\eta(X)Y-a(Z)a(X)Y$$
$$+\,a(Z)\eta(Y)X+a(Z)a(Y)X$$

Bianchi first identity for a linear connection on M^n is given by (Sinha 1982)

$$(6.4)\quad \overline{R}(X,Y)Z+\overline{R}(Y,Z)X+\overline{R}(Z,X)Y=\overline{T}(\overline{T}(X,Y),Z)+\overline{T}(\overline{T}(Y,Z),X)$$
$$+\overline{T}(\overline{T}(Z,X),Y)+\ (\overline{D}_X\overline{T})(Y,Z)+(\overline{D}_Y\overline{T})(Z,X)+(\overline{D}_Z\overline{T})(X,Y).$$

Using (6.2) and (6.3) and(6.4) we get

$$(6.5)\quad \overline{R}(X,Y)Z+\overline{R}(Y,Z)X+\overline{R}(Z,X)Y=da(X,Y)Z+da(Y,Z)X+da(Z,X)Y$$
$$+\,a(X)\eta(Z)Y-a(X)\eta(Y)Z+a(Y)\eta(X)Z$$
$$-\,a(Y)\eta(Z)X+a(Z)\eta(Y)X-a(Z)\eta(X)Y.$$

We call (6.5) as the first Bianchi's identity with respect to semi-symmetric non-metric connection $\overline{D}$ in an LP-cosympletic manifold.

7. Einstein Manifold with respect to semi-symmetric non-metric connection on LP-cosympletic manifold

A Riemannian manifold Mn is called an Einstein manifold with respect to Riemannian connection if

$$(7.1)\qquad S(X,Y)\ =\ \frac{r}{n}g(X,Y).$$

Analogous to this definition, we define Einstein manifold with respect to semi-symmetric non metric connection $\overline{D}$

$$(7.2)\qquad \overline{S}(X,Y)=\frac{\overline{r}}{n}\,g(X,Y).$$

From (4.4) ,(4.5) and(7.2) we have

$$\overline{S}(X,Y)-\frac{\overline{r}}{n}\,g(X,Y)=S(Y,Z)+(n-1)\eta(Y)\eta(Z)-da(Z,Y)-\frac{r+1-n+\lambda}{n}g(X,Y)$$

$$(7.3)\qquad \overline{S}(X,Y)-\frac{\overline{r}}{n}g(X,Y)=S(Y,Z)-\frac{r}{n}g(X,Y)+(n-1)\eta(Y)\eta(Z)$$

$$-da(Z,Y) + \frac{\lambda + 1 - n}{n} g(X,Y).$$

If

$$n(n-1)\eta(Y)\eta(Z) + (\lambda + 1 - n)\, g(X,Y) = n.da(Z,Y) \tag{7.4}$$

then from (7.3), we get

$$\overline{S}(X,Y) - \frac{\overline{r}}{n} g(X,Y) = S(X,Y) - \frac{r}{n} g(X,Y).$$

Hence we have

Theorem (7.1) : If the relation(7.4) holds in an LP-cosympletic manifold M^n with semi-symmetric non metric connection, then the manifold is an Einstein manifold for the Riemannian connection if and only if it is an Einstein manifold for the connection $\overline{D}$.

8. Weyl Projective Curvature Tensor

If $\overline{P}$ and P denote the projective curvature tensor with respect to $\overline{D}$ and D respectively, then we have

$$\overline{P}(X,Y)Z = \overline{R}(X,Y)Z - \frac{1}{n-1}[\overline{S}(Y,Z)X - \overline{S}(X,Z)Y] \tag{8.1}$$

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y] \tag{8.2}$$

Using (4.4) and (4.3) in equation (8.1), we have

$$\begin{aligned} \overline{P}(X,Y)Z &= R(X,Y,Z) + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + da(X,Y)Z \\ &\quad - \frac{1}{n-1}[S(Y,Z)X + (n-1)\eta(Y)\eta(Z)X - da(Z,Y)X - S(X,Z)Y \\ &\qquad - (n-1)\eta(X)\eta(Z)Y - da(X,Z)Y] \\ &= R(X,Y,Z) - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y] + \frac{1}{n-1}[(n-1)da(X,Y)Z - \\ &\qquad da(Z,Y)X + da(X,Z)Y]. \end{aligned}$$

(8.3)

$$\overline{P}(X,Y)Z = P(X,Y)Z + \frac{1}{n-1}[(n-1)da(X,Y)Z - da(Z,Y)X + da(X,Z)Y]$$

It is clear that if 1-form a is closed i.e. $da = 0$. Then from (8.3) we get

$$\overline{P}(X,Y)Z = P(X,Y)Z.$$

Hence we have

Theorem (8.1) : If in an LP-cosympletic manifold M^n admits a semi-symmetric non metric connection $\overline{D}$ then the Weyl projective curvature tensor of $\overline{D}$ is equal to the Weyl projective tensor of D if 1-form is closed.

$$\overline{P}(X,Y)Z = 0 \tag{8.4}$$

Which implies $\overline{S}(Y,Z) = 0$.

Then from (8.3), we have

$$P(X,Y)Z = \frac{1}{n-1}[da(Z,Y) - (n-1)da(X,Y)Z - da(X,Z)Y]. \tag{8.5}$$

If 1-form a is closed i.e. $da = 0$.

Then from (8.5) we get

$$P(X,Y)\,Z = 0.$$

Hence we have

Theorem (8.2) : If in an Lorentzian Para cosympletic manifold M^n the curvature tensor of semi-symmetric non-metric connection $\overline{D}$ vanish and 1-form a is closed, then the manifold is projectively flat.

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