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Study on Einstein-Käehlerian Decomposable Recurrent Space of First Order

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Abstract

Takano [2] have studied decomposition of curvature tensor in a recurrent space. Sinha and Singh [3] have been studied and defined decomposition of recurrent curvature tensor field in a Finsler space. Singh and Negi studied decomposition of recurrent curvature tensor field in a Käehlerian space. Negi and Rawat [6] have studied decomposition of recurrent curvature tensor field in Käehlerian space. Rawat and Silswal [11] studied and defined decomposition of recurrent curvature tensor fields in a Tachibana space. Further, Rawat and Kunwar Singh [12] studied the decomposition of curvature tensor field in Käehlerian recurrent space of first order.

In the present paper, we have studied the decomposition of curvature tensor fields R^h_{ijk} in terms of two non-zero vectors and a tensor field in Einstein-Käehlerian recurrent space of first order and several theorem have been established and proved.

Keywords : Käehlerian space, Einstein space, Einstein-Käehlerian space, recurrent space, Curvature tensor, Projective curvature tensor.

1. Introduction

An $n(= 2m)$ dimensional Käehlerian space K_n is a Riemannian space, which admits a tensor field F^h_i satisfying the conditions

$$F^h_i \, F^j_h = -\delta^j_i \tag{1.1}$$

$$F_{ij} = -F_{ji}, \quad (F_{ij} = F^a_i \, g_{aj}) \tag{1.2}$$

and

$$F^h_{i,j} = 0, \tag{1.3}$$

where the (,) followed by on index denotes the operation of covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space.

The Riemannian curvature tensor R^h_{ijk} is given by

$$R^h_{ijk} = \partial_i \begin{Bmatrix} h \\ j\ k \end{Bmatrix} - \partial_j \begin{Bmatrix} h \\ i\ k \end{Bmatrix} + \begin{Bmatrix} h \\ i\ \alpha \end{Bmatrix} \begin{Bmatrix} \alpha \\ j\ k \end{Bmatrix} - \begin{Bmatrix} h \\ j\ \alpha \end{Bmatrix} \begin{Bmatrix} \alpha \\ i\ k \end{Bmatrix},$$

The Ricci tensor and the scalar Curvature tensor are respectively given by

$$R_{ij} = R^a_{aij} \text{ and } R = g_{ij}\, R_{ij}. \tag{1.4}$$

It is well known that these tensors satisfies the following identities

$$R^a_{ijk} = R_{jk,i} - R_{ik,j}, \tag{1.5}$$

$$R_{,i} = 2R^a_{i,a}, \tag{1.6}$$

$$F^a_i R_{aj} = -R_{ia} F^a_j, \tag{1.7}$$

and

$$F^a_i R^i_a = R^a_i F^i_a. \tag{1.8}$$

The holomorphically projective curvature tensor P^h_{ijk} is defined by

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{(n+2)}(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}S^h_i + 2S_{ij}F^h_k), \tag{1.9}$$

where $\quad S_{ij} = F^a_i R_{aj}.$

Let us suppose that a Käehlerian space is Einstein one, and then the Ricci tensor satisfies

$$R_{ij} = \frac{R}{n}\, g_{ij}, \qquad R_{,a} = 0$$

from which, we obtain

$$R_{ij}, a = 0, \qquad S_{ij,a} = 0$$

and $\quad S_{ij} = \frac{R}{n}\, F_{ij}.$

The Bianchi identity for Einstein-Käehlerian space are given by

$$R^h_{ijk} + R^h_{jki} + R^h_{kij} = 0, \tag{1.10}$$

and

$$R^h_{ijk,a} + R^h_{ika,j} + R^h_{iaj,k}. \tag{1.11}$$

The Commutative formulae for the Curvature tensor fields are given as follows

$$T^i_{,jk} - T^i_{,kj} = T^a\, R^i_{ajk}, \tag{1.12}$$

$$T^h_{i,ml} - T^h_{i,lm} = T^a_i R^h_{aml} - T^h_a R^a_{iml}. \tag{1.13}$$

A Einstein-Käehlerian space is said to be Einstein-Käehlerian recurrent space of first order, if its curvature tensor field satisfy the condition

$$R^h_{ijk,a} = \lambda_a R^h_{ijk} \tag{1.14}$$

where λ_a is a non-zero vector and is known as recurrence vector field.

The following relations follow immediately from equation (1.14),

$$R_{ij,a} = \lambda_a R_{ij} \tag{1.15}$$

and

$$R_{,a} = \lambda_a R. \tag{1.16}$$

1. Decomposition of Curvature Tensor Field R^h_{ijk}

We Consider the decomposition of recurrent curvature tensor field R^h_{ijk} in the following form

$$R^h_{ijk} = V^h_i \phi_j \psi_k \tag{2.1}$$

where the non-zero tensor field V^h_i and vector ϕ_j, ψ_k are such that

$$\lambda_h V^h_i = P_i. \tag{2.2}$$

Theorem (2.1): Under the decomposition (2.1), the Bianchi identities for R^h_{ijk} takes the forms

$$P_i \phi_j \psi_k + P_j \phi_k \psi_i + P_k \phi_i \psi_j = 0 \tag{2.3}$$

and

$$\lambda_a \phi_j \psi_k + \lambda_j \phi_k \psi_a + \lambda_k \phi_a \psi_j = 0. \tag{2.4}$$

Proof. From Equations (1.10) and (2.1), we have

$$V^h_i \phi_j \psi_k + V^h_j \phi_k \psi_i + V^h_k \phi_i \psi_j = 0. \tag{2.5}$$

Multiplying (2.5), by λ_h, and using (2.2), we get relation (2.3)

$$P_i \phi_j \psi_k + P_j \phi_k \psi_i + P_k \phi_i \psi_j = 0.$$

From Equations (1.11), (1.14) and (2.1), we have

$$V^h_i [\lambda_a \phi_j \psi_k + \lambda_j \phi_k \psi_a + \lambda_k \phi_a \psi_j] = 0. \tag{2.6}$$

Multiplying (2.6) by λ_h and using (2.2), we get relation (2.4).

Theorem (2.2): Under the decomposition (2.1), the tensor field R^h_{ijk}, R_{ij} and vectors ϕ_j, ψ_k satisfies the relations

$$\lambda_a R^a_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik} = P_i \phi_j \psi_k. \tag{2.7}$$

Proof. With the help of Equations (1.5), (1.14) and (1.15), we have

$$\lambda_a R^a_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik}. \tag{2.8}$$

Multiplying (2.1) by λ_h, and using relation (2.2), we get

$$\lambda_h R^h_{ijk} = P_i \phi_j \phi_k \tag{2.9}$$

in view of (2.8) and (2.9), we get the required relation (2.7).

Theorem (2.3): Under the decomposition (2.1), the quantities λ_a and V_i^h behave like the recurrent vector and tensor field.

The recurrent form of these quantities are given by

$$\lambda_{a,m} = \mu_m \lambda_a, \tag{2.10}$$

$$V^h_{i,m} = \mu_m V_i^h. \tag{2.11}$$

Proof. Differentiating (2.7), covariantly with respect to x^m, and using (2.1) and (2.7), we obtain

$$\lambda_{a,m} V_i^a \phi_j \psi_k = \lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}. \tag{2.12}$$

Multiplying (2.12) by λ_a and using (2.1) and (2.8), we get

$$\lambda_{a,m}(\lambda_i R_{jk} - \lambda_j R_{ik}) = \lambda_a(\lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}). \tag{2.13}$$

Now, multiplying (2.13) by λ_h, we have

$$\lambda_{a,m}(\lambda_i R_{jk} - \lambda_j R_{ik})\lambda_h = \lambda_a \lambda_h(\lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}. \tag{2.14}$$

Since the expression of right hand side of the above equation is symmetric in a and h, therefore

$$\lambda_{a,m}\lambda_h = \lambda_{h,m}\lambda_a, \tag{2.15}$$

provided that $\lambda_i R_{jk} - \lambda_j R_{ik} \neq 0.$

The vector field λ_a being non-zero, we can have a proportional vector μ_m such that

$$\lambda_{a,m} = \mu_m \lambda_a. \tag{2.16}$$

Further, differentiating (2,2) w.r. to x^m and using (2.16), we get

$$\lambda_h V^h_{i,m} = P_{i,m} - \mu_m P_i \tag{2.17}$$

from the above equation, it is obvious that

$$\lambda_h V^h_{i,m} = \lambda_a V^a_{i,m}. \tag{2.18}$$

Since λ_a is a non-zero recurrence vector field, we can get a proportional vector field μ_m such that

$$V^h_{i,m} = \mu_m V^h_i$$

which complete the proof.

Theorem (2.4): Under the decomposition (2.1), the vector field P_i, ϕ_j, ψ_k behave like recurrent vectors and their recurrent form are given respectively by

$$P_{i,m} = 2\mu_m P_i \tag{2.19}$$

and

$$(\lambda_m - \mu_m)\phi_j\psi_k = \phi_{j,m}\psi_k + \phi_j\psi_{k,m}. \tag{2.20}$$

Proof. Differentiating (2.2) covariantly w.r. to x^m, and using equation (2.2), (2.10) and (2.11), we obtain the required result (2.19). Further, differentiating equation (2.1) covariantly w.r. to x^m and using equation (1.14), (2.1) and (2.11), we get the required recurrent form (2.20).

Theorem (2.5): Under the decomposition (2.1), the curvature tensor and holomorphically projective curvature tensor are equal if

$$\phi_k\psi_l\{(P_i\delta^h_j - P_j\delta^h_i) + P_a(F^a_i F^h_j - F^a_j F^h_i)\} + 2P_a\phi_j\psi_l F^a_i F^h_k = 0. \tag{2.21}$$

Proof. The equation (1.9), may be written in the form

$$P^h_{ijk} = R^h_{ijk} + D^h_{ijk} \tag{2.22}$$

where

$$D^h_{ijk} = \frac{1}{n+2}(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}F^h_i + 2S_{ij}F^h_k). \tag{2.23}$$

Contracting indices h and k in (2.1), we obtain

$$R_{ij} = V^l_i\phi_j\psi_l. \tag{2.24}$$

In view of equation (2.24), we have

$$S_{ij} = F^a_i\phi_j\psi_l V^l_a. \tag{2.25}$$

Making use of (2.24) and (2.25) in equation (2.22), we get

$$D^h_{ijk} = \frac{1}{n+2}[\phi_k\psi_l\{(V^l_i\delta^h_j - V^l_j\delta^h_i) + V^l_a(F^a_i F^h_j - F^a_j F^h_i)\} + 2\phi_j\psi_l F^a_i F^h_k V^l_a]. \tag{2.26}$$

In view of (2.23), it is clear that

$$P^h_{ijk} = R^h_{ijk}$$

iff $D^h_{ijk} = 0$, which in view of equation (2.26) gives

$$[\phi_k\psi_l\{(V^l_i\delta^h_j - V^l_j\delta^h_i) + V^l_a(F^a_iF^h_j - F^a_jF^h_i)\} + 2\phi_j\psi_lF^a_iF^h_kV^l_a] = 0. \qquad (2.27)$$

Multiplying (2.27) by λ_l and using (2.2), we obtain the required condition (2.21).

Theorem (2.6): Under the decomposition (2.1), the scalar curvature R, satisfy the relation

$$\lambda_kR = g^{ij}P_i\phi_j\psi_l. \qquad (2.28)$$

Proof. Contracting indices h and k in (2.1), we get

$$R_{ij} = V^l_i\phi_j\psi_l. \qquad (2.29)$$

Multiplying (2.29) by g^{ij} both sides, we get

$$g^{ij}R_{ij} = g^{ij}V^l_i\phi_j\psi_l \qquad (2.30)$$

in view of Equation (1.4), the above equation reduces to

$$R = g^{ij}V^l_i\phi_j\psi_l. \qquad (2.31)$$

Now multiplying (2.31) by λ_l and using (2.2), we get

$$\lambda_lR = g^{ij}P_i\phi_j\psi_l$$

which complete the proof of the theorem.

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