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Additive Analogue of Certain Arithmetical Functions

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Abstract

J. Sandor and E. Egri [4] have defined an arithmetic function related to Euler minimum function which has been defined by J. Sandor [3]. We discuss some particular cases of this arithmetic function for some certain arithmetical functions.

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1. Introduction

J. Sandor [2] have introduced a function which is defined as

$$F_f^A(n) = \min\{k \epsilon A : n/f(k)\}, \tag{1.1}$$

where $A \subset N$, and $f : N \to N$ be an arithmetic function.

Now, J. Sandor [3], for $f(k) = \phi(k)$,Euler totient function and $A = N$, Euler minimum function has introduced, which is defined as

$$E(n) = \min\{k \epsilon N : n/\phi(k)\}. \tag{1.2}$$

J. Sandor [1], have studied the particular case of (1.1) for $f(k) = \phi^*(k)$, unitary totient function and called as unitary totient minimum function defined as

$$E^*(n) = \min\{k \geq 1 : n/\phi^*(k)\}. \tag{1.3}$$

Recently an arithmetic function related to Euler minimum function have been introduced in J. Sandor and E. Egri [4] defined as

$$H_\phi(n) = \min\{k \geq 1 : \phi(n)/\phi(k)\}, \tag{1.4}$$

and more generally,

$$H_g(n) = \min\{k \geq 1 : g(n)/g(k)\}, \tag{1.5}$$

for a given arithmetic function $g : N \to N$. In this paper we study the arithmetic function given in (1.5) for $g(n) = \phi^*(n)$, and $g(n) = R(n)$, product of divisors of n, where unitary totient function $\phi^*(n)$ is defined as

$$\phi^*(n) = \begin{cases} (p_1^{\alpha_1} - 1)(p_2^{\alpha_2} - 1).....(p_r^{\alpha_r} - 1) & , \quad n = \prod_{i=1}^{r} p_i^{\alpha_i} \\ 1 & , \quad n = 1 \end{cases} \tag{1.6}$$

and product of divisor function $R(n)$ is defined as

$$R(n) = d_1.d_2....d_r, \tag{1.7}$$

where $d_1, d_{2,}..., d_r$ are divisors of n. Also,

$$R(n) = (n)^{\frac{d(n)}{2}}, \tag{1.8}$$

where $d(n)$ denotes number of divisor of n.

In analogy with (1.5), we can define

$$H_{\phi^*}(n) = \min\{k \geq 1 : \phi^*(n)/\phi^*(k)\}, \tag{1.9}$$

and

$$H_R(n) = \min\{k \geq 1 : R(n)/R(k)\}. \tag{1.10}$$

We prove some important results related with (1.9) and (1.10) in section 2.

2. Important Results

Theorem 2.1. (a) $H_{\phi^*}(p^\alpha) = \begin{cases} 1 & , \quad p = 2 \text{ with } \alpha = 1 \\ p^\alpha & , \quad p = 2 \text{ with } \alpha > 1 \\ p^\alpha & , \quad p \geq 3 \text{ with } \alpha \geq 1 \end{cases}$,

(b) $H_{\phi^*}(2p^\alpha) = \begin{cases} 2^{\alpha+1} & , \quad p = 2, \text{with } \alpha > 0 \\ p^\alpha & , \quad p \geq 3 \text{ with } \alpha \geq 1 \end{cases}$,

(c) If n is odd then $H_{\phi^*}(2n) = H_{\phi^*}(n)$.

Proof.(a) Using (1.9), It is clear that $H_{\phi^*}(n) \leq n$ as $\phi^*(n)/\phi^*(n)$.

Let $\phi^*(p^\alpha)/\phi^*(k)$ then using (1.6), we have

$$(p^\alpha - 1) = \phi^*(p^\alpha) \leq \phi^*(k) \leq k - 1 \quad \text{for} \quad k \geq 2$$

so,

$$p^\alpha \leq k.$$

This implies that

$$H_{\phi^*}(p^\alpha) \geq p^\alpha.$$

Now, for $p^\alpha = 2$ it is clear that $H_{\phi^*}(p^\alpha) = 1$.

For $p = 2$, with $\alpha > 1$ and $p \geq 3$, with $\alpha \geq 1$, $H_{\phi^*}(p^\alpha) = p^\alpha$.
If n is odd then

$$\phi^*(2n) = \phi^*(n),$$

so (c) follows. This implies also (b).

Theorem 2.2.

(a) $\sqrt{n} \leq H_{\phi^*}(n) \leq n$ for $n \geq 6$,

(b) $H_{\phi^*}(n) = H_{\phi^*}(m)$ if $\phi^*(n) = \phi^*(m)$.

Proof. (a) Using (1.9), $H_{\phi^*}(n) \leq n$ as $\phi^*(n)/\phi^*(n)$, and $H_{\phi^*}(n) \geq \phi^*(n), n \geq 1$.

If $\phi^*(n)/\phi^*(k)$, then

$$\phi^*(n) \leq \phi^*(k) \leq k - 1 \quad \text{for} \quad k \geq 2.$$

Since, $\phi(n) > \sqrt{n}$ for $n \geq 6$ and $\phi(n) \leq \phi^*(n)$,therefore,

$$\phi^*(n) > \sqrt{n} \text{ for } n \geq 6.$$

So (a) is proved.

(b)

$$\begin{aligned} H_{\phi^*}(n) &= \min\{k \geq 1 : \phi^*(n)/\phi^*(k)\} \\ &= \min\{k \geq 1 : \phi^*(m)/\phi^*(k)\} \\ &= H_{\phi^*}(m), \end{aligned}$$

if $\phi^*(n) = \phi^*(m)$.

Theorem 2.3. If $H_{\phi^*}(m)/H_{\phi^*}(n)$, then $[\phi^*(m), \phi^*(n)]/\phi(H_{\phi^*}(n))$, where [ ] dentoes L.C.M.

Proof. Let $x = H_{\phi^*}(m)$ and $y = H_{\phi^*}(n)$. Thus using (1.6), $\phi^*(m)/\phi^*(x)$ and $\phi^*(n)/\phi^*(y)$.

Now it is given that x/y so

$$\phi^*(x)/\phi^*(y),$$

this implies that

$$\phi^*(m)/\phi^*(x)/\phi^*(y),$$

giving

$$\phi^*(m)/\phi^*(y).$$

Hence

$$[\phi^*(m), \phi^*(n)]/\phi^*(y).$$

Theorem 2.4. If $H_{\phi^*}(m)/H_{\phi^*}(n)$, then

$$\phi^*(m)/(\phi^*(H_{\phi^*}(m)), \phi^*(H_{\phi^*}(n)).$$

Proof. Let $H_{\phi^*}(m) = x$ and $H_{\phi^*}(n) = y$, then it is given that x/y, this implies that

$$\phi^*(x)/\phi^*(y),$$

but

$$\phi^*(m)/\phi^*(x)/\phi^*(y),$$

so,

$$\phi^*(m)/\phi^*(y).$$

Thus

$$\phi^*(m)/(\phi^*(x), \phi^*(y)).$$

Theorem 2.5. If $H\phi^*(m)/H\phi^*(n)$, then

$$\phi^*(m) \leq |\phi^*(H_{\phi^*}(n)) - \phi^*(H_{\phi^*}(m))|.$$

Proof. Since, $(a, b) \leq |b - a|$ for $a \neq b$, therefore, by theorem 2.4,

$$\phi^*(m) \leq (\phi^*(x), \phi^*(y)) \leq |\phi^*(H_{\phi^*}(n)) - \phi^*(H_{\phi^*}(m))|.$$

Theorem 2.6. (a) $H_R(n) = n \quad \forall n \geq 1,$
(b) $H_R(2n) = 2.H_R(n) \quad \forall n \geq 1,$
(c) $H_R(n^2) = n.H_R(n) \quad \forall n \geq 1,$
(d) $H_R(n) = H_R(m)$ if $R(n) = R(m)$.

Proof. Using (1.10), clearly $H_R(n) \leq n$ as $R(n)/R(n)$.On the other hand, since $k/R(k)$ therefore, $R(n) \leq k$ as $n \leq R(n)$, thus n$\leq k$.This gives $H_R(n) \geq n$.

Hence $H_R(n) = n$ $\forall n \geq 1$.Since, $H_R(2n) = 2n$ (using (a)) therefore, $H_R(2n) = 2.H_R(n)$. Hence (b) is proved. Using (a) we get, $H_R(n^2) = n^2 = n.H_R(n)$. Hence (c) is proved.

Now (d) is followed by (1.10) as

$$\begin{aligned} H_R(n) &= \min\{k \geq 1 : R(n)/R(k)\} \\ &= \min\{k \geq 1 : R(m)/R(k)\} \\ &= H_R(m). \end{aligned}$$

Hence (d) is proved.

Theorem 2.7. If $R(m)/R(n)$, then $(R(m), R(n)) = H_R(R(m))$, where () denotes *g.c.d* of $R(m)$ and $R(n)$.

Proof. Since $R(m)/R(n)$ therefore, $(R(m), R(n)) = R(m)$, using theorem 2.6(a), we get $H_R(R(m)) = R(m)$.

Hence $(R(m), R(n)) = H_R(R(m))$.

Theorem 2.8. If $R(m)/R(n)$, then $[R(m), R(n)] = H_R(R(n))$, where [ ] denotes L.C.M. of $R(m)$and $R(n)$.

Proof. Since, $R(m)/R(n)$ therefore, $[R(m), R(n)] = R(n)$, using theorem 2.6(a), we get

$$H_R(R(n)) = R(n) = [R(m), R(n)].$$

Hence theorem is proved.

Theorem 2.9. *If* $H_R(m)/H_R(n)$, *then* $[R(m), R(n)]/R(H_R(n))$.

Proof. Let $x = H_R(m)$ and $y = H_R(n)$, using (1.10), $R(m)/R(x)$and $R(n)/R(y)$. On the other hand if x/y then $R(x)/R(y)$. So, using theorem 2.6(a) $R(m)/R(n)/R(y)$ this implies $R(m)/R(y)$. But $R(n)/R(y)$ too. So, $[R(m), R(n)]/R(y)$. Hence $[R(m), R(n)]/R(H_R(n))$.

Theorem 2.10. If $R(m)/R(n)$, then $H_R(R(m)) \leq |R(n) - R(m)|$.

Proof. Using theorem 2.7, if $R(m)/R(n)$ then, $(R(m), R(n)) = H_R(R(m))$.Since, $(a, b) \leq |b - a|$,therefore, $(R(m), R(n)) = H_R(R(m)) \leq |R(n) - R(m)|$.

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