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On APST-Riemannian Manifold-I

Amar Nath and Manohar Singh Chahar

Department of Mathematics
College of Engineering Roorkee, Roorkee, India
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Abstract

In this paper, Almost para sasakian type-Riemannian manifold have been studied. The first section is introductory. Basic definition and known results are defined. Second section deals with APST-Riemannian manifold and the third section is devoted for PKCT-Riemannian manifold. Some interesting results have been investigated.

1. Introduction

Definition (1.1): Let an n dimensional Riemannian manifold M_n, on which there are defined a tensor field F of type (1,1), a vector field T, a $1-$form A and metric tensor g satisfying for arbitrary vector field X, Y, $Z, \ldots$ satisfying,

$$\begin{array}{ll} \text{(a)} & F^2X = X - A(X)T, \\ \text{(b)} & F(T) = 0, \\ \text{(c)} & A(FX) = 0, \\ \text{(d)} & A(T) = 1, \\ \text{(e)} & g(Fx, Fy) = -g(x, y) + A(x)A(y). \end{array} \tag{1.1}$$

Then structure (F, T, A, g) is called almost para contact metric structure and manifold Mn will be called Almost para contact metric Riemannian manifold.

Let us put

(f) $'F(x, y) = g(\overline{X}, y)$

where $\overline{X} = Fx$. We can verify that $'F$ is skew symmetric, i.e.,

$$'F(x, y) + 'F(y, x) = 0,$$

one can check from (1.1) (e), that

(g) $g(T, X) = A(X)$. [1]

Definition (1.2): An almost para contact metric manifold on which the fundamental 2-form $'F$ satisfies

$$2\,'F = dA \tag{1.2}$$

is called an almost para sasakian type manifold (APST-Riemannian manifold) or contact Riemannian manifold. [2]

2. Some Properties of APST-Riemannian Manifold

We have from (1.2)

$$\begin{aligned} 2\,'F &= dA, \\ \text{or} \qquad 2\,'F(X,Y) &= dA(X,Y), \\ &= XA(Y) - YA(X) - A([X,Y]), \\ &= (D_xA)(Y) - (D_yA)(X), \end{aligned}$$

where 'D' is Riemannian connexion. Thus we have

Theorem (2.1): On APST - Riemannian manifold, we have

$$'F(X,Y) = \frac{1}{2}\,[(D_xA)(Y) - (D_yA)(X)]. \tag{2.1}$$

We have from (1.2)

$$(d\,'F) = d^2A = 0 \tag{2.2}$$

$$\begin{aligned} &(d\,'F)(X,Y,Z) = X\,'F(Y,Z) - Y\,'F(X,Z) + Z\,'F(X,Y) \\ &- \,'F([X,Y],Z) + \,'F([X,Z],Y) - \,'F([Y,Z],X) = D_x\,'F(Y,Z) \qquad (2.3) \\ &+ \,'F(D_xY,Z) + \,'F(Y,D_xZ) - (D_y\,'F)(X,Z) - \,'F(D_yX,Z) \end{aligned}$$

$$\begin{aligned} &- \,'F(X,D_yZ) + (D_z\,'F)(X,Y) + \,'F(D_zX,Y) + \,'F(X,D_zY) \\ &- \,'F((D_xY - D_yX),Z) + \,'F((D_xZ - D_zX),Y) - \,'F((D_yZ - D_zY),X) \\ &= (D_x\,'F)(Y,Z) + (D_y\,'F)(Z,X) + (D_z\,'F)(X,Y). \end{aligned} \tag{2.4}$$

Thus we have

Theorem (2.2): On APST-Riemannian manifold, we have

$$(d\,'F) = 0 \text{ (i.e. } 'F \text{ is closed)}$$

$$\Leftrightarrow (D_x\,'F)(Y,Z) + (D_y\,'F)(Z,X) + (D_z\,'F)(X,Y) = 0. \tag{2.5}$$

Definition (2.1): An almost para contact metric manifold on which $'F$ is closed is called Para Quasi-Sasakian type Manifold or in short PQST manifold.

Definition (2.2): An APST-Riemannian manifold, on which

$$(D_xA)(Y) + (D_yA)(X) = 0 \tag{2.6}$$

holds, is called para$-K-$contact type Riemannian manifold or PKCT-Riemannian manifold. [2]

3. Theorems on PKCT-Riemannian manifold

From (2.1) and (2.6)

$$2\,'F(X,Y) + 0 = [(D_xA)(Y) - (D_yA)(X)] + [(D_xA)(Y) + (D_yA)(X)]$$

$$2\,'F(X,Y) = 2(D_xA)(Y)$$

$$'F(X,Y) = (D_xA)(Y) = -(D_yA)(X).$$

Thus we have

Theorem (3.1): On PKCT-Riemannian manifold, we have

$$'F(X,Y) = (D_xA)(Y) = -(D_yA)(X). \tag{3.1}$$

From (3.1)

$$'F(X,Y) = (D_xA)(Y)$$

$$(D_z\,'F)(X,Y) + \,'F(D_zX,Y) + \,'F(X,D_zY) = (D_zD_xA)(Y) + (D_xA)(D_zY)$$

$$(D_z\,'F)(X,Y) + (D_{D_zx}A)(Y) + (D_xA)(D_zY) = (D_zD_xA)(Y) + (D_xA)(D_zY)$$

$$(D_z\,'F)(X,Y) = (D_zD_xA)(Y) - (D_{D_zx}A)(Y) \tag{3.2}$$

$$(D_x\,'F)(Y,Z) = (D_xD_yA)(Z) - (D_{D_xy}A)(Z) \tag{3.3}$$

$$(D_y\,'F)(Z,X) = (D_yD_zA)(X) - (D_{D_yz}A)(X) \tag{3.4}$$

Replace Z by X

$$(D_y\,'F)(X,Z) = (D_yD_xA)(Z) - (D_{D_yx}A)(Z) \tag{3.5}$$

Subtracting (3.5) from (3.3), we get

$$\begin{aligned}(D_x\,'F)(Y,Z) - (D_y\,'F)(X,Z) =& (D_xD_yA)(Z) - (D_yD_xA)(Z) - (D_{D_xy}A \\ &- D_{D_yx}A)(Z)\end{aligned} \tag{3.6}$$

$$(D_x\,'F)(Y,Z) + (D_y\,'F)(Z,X) = (D_xD_yA)(Z) - (D_yD_xA)(Z) - (D[X,Y]A)(Z). \tag{3.7}$$

Using (2.5), we get

$$-(D_z\,'F)(X,Y) = -A(K(X,Y,Z)),$$

$$(D_z\,'F)(X,Y) = A(K(X,Y,Z)).$$

Thus we have

Theorem (3.2): On PKCT-Riemannian manifold, we have

$$(D_z\, 'F)(X,Y) = A(K(X,Y,Z)). \tag{3.8}$$

We have from (1.1)(d)

$$A(T) = 1$$

$$(D_x A)T + A(D_x T) = 0.$$

Using (3.1) we get

$$'F(X,T) + A(D_x T) = 0.$$

Using (1.1)(f), we get

$$g(\overline{X},T) + A(D_x T) = 0,$$

$$A(\overline{X}) + A(D_x T) = 0,$$

$$A(\overline{X} + D_x T) = 0,$$

$$D_x T = -\,\overline{X}.$$

Thus we have

Theorem (3.3): On PKCT-Riemannian manifold, we have

$$D_x T = -\,\overline{X}. \tag{3.9}$$

Alternative definition of PKCT-Riemannian manifold is given by

Definition (3.1): An almost para contact metric type Riemannian manifold on which

$$D_x T = -\,\overline{X}$$

is called PKCT-Riemannian manifold.

We have from (1.1)(f) and (1.1)(e)

$$\begin{aligned} &'F(Y,T) = g(\overline{Y},T) = A(\overline{Y}) = 0 \\ &'F(Y,T) = 0 \\ &(D_x\, 'F)(Y,T) + \,'F(D_x Y,T) + \,'F(Y,D_x T) = 0 \\ &(D_x\, 'F)(Y,T) + 0 - \,'F(D_x T,Y) = 0. \end{aligned} \tag{3.10}$$

Using (3.9), we get

$$\begin{aligned} (D_x\, 'F)(Y,T) &= \,'F(-\overline{X},Y) \\ (D_x\, 'F)(Y,T) &= -\,'F(\overline{X},Y) \\ (D_x\, 'F)(Y,T) &= \,'F(Y,\overline{X}). \end{aligned} \tag{3.11}$$

Using (1.1)(f)

$$(D_x\,'F)(Y,T) = g(\overline{Y X}),$$

$$(D_x\,'F)(Y,T) = g(\overline{X},\overline{Y}).$$

Thus we have

Theorem (3.4): On PKCT-Riemannian manifold, we have

$$D_x\,'F(Y,T) = g(\overline{X},\overline{Y}). \tag{3.12}$$

We have from (1.1)(f)

$$\begin{aligned}
'F(X,Y) &= g(\overline{X},Y)\\
'F(\overline{X},\overline{Y}) &= g(\overline{\overline{X}},\overline{Y})\\
'F(\overline{X},\overline{Y}) &= g(X - A(X)T,\overline{Y})\\
'F(\overline{X},\overline{Y}) &= g(X,\overline{Y}) - A(X)g(T,\overline{Y})\\
'F(\overline{X},\overline{Y}) &= g(X,\overline{Y}) - A(X)A(\overline{Y})\\
'F(\overline{X},\overline{Y}) &= g(X,\overline{Y})\\
'F(\overline{X},\overline{Y}) &= {'F}(Y,X)\\
'F(\overline{X},\overline{Y}) &= -\,'F(X,Y).
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
&(D_z\,'F)(\overline{X},\overline{Y}) + {'F}((D_zF)(X) + F(D_zX),\overline{Y}) + {'F}(\overline{X},(D_zF)(Y)\\
&+ F(D_zY)) = -(D_z\,'F)(X,Y) - {'F}(D_zX,Y) - {'F}(X,D_zY)\\
&(D_z\,'F)(\overline{X},\overline{Y}) + {'F}((D_zF)(X),\overline{Y}) + {'F}(F(D_zX),\overline{Y}) + {'F}(\overline{X},(D_zF)(Y))\\
&+ {'F}(\overline{X},F(D_zY)) = -(D_z\,'F)(X,Y) - {'F}(D_zX,Y) - {'F}(X,D_zY).
\end{aligned} \tag{3.14}$$

Using (3.13), we get

$$(D_z\,'F)(\overline{X},\overline{Y}) + {'F}((D_zF)(X),\overline{Y}) + {'F}(\overline{X},(D_zF)(Y)) = -(D_z\,'F)(X,Y). \tag{3.15}$$

Using (1.1)(f), we get

$$(D_z\,'F)(\overline{X},\overline{Y}) + g((\overline{D_zF)\,(X}),\overline{Y}) - g((\overline{D_zF)(Y})\overline{X}) = -(D_z\,'F)(X,Y). \tag{3.16}$$

Using (1.1)(e), we get

$$\begin{aligned}
&(D_z\,'F)(\overline{X},\overline{Y}) - g((D_zF)(X),Y) + A((D_zF)(X))A(Y)\\
&+ g((D_zF)(Y),X) - A((D_zF)(Y)).A(X) = -(D_z\,'F)(X,Y)
\end{aligned} \tag{3.17}$$

$$(D_z\,'F)(\overline{X},\overline{Y}) + (D_z\,'F)(Y,X) + A((D_zF)(X))A(Y) - A((D_zF)(Y))A(X) = 0 \tag{3.18}$$

$$(D_z\,'F)(\overline{X},\overline{Y}) + (D_z\,'F)(Y,X) + g(\overline{Z},\,\overline{X})A(Y) - g(\overline{Z},\,\overline{Y})A(X) = 0. \tag{3.19}$$

Using (1.1)(e), we get

$$(D_z\,'F)(\overline{X},\overline{Y}) - (D_z\,'F)(X,Y) - g(Z,X)A(Y) + g(Z,Y)A(X) = 0.$$

Thus we have

Theorem (3.5): On PKCT-Riemannian manifold, we have

$$(D_z\,'F)(\overline{X},\overline{Y}) - (D_z\,'F)(X,Y) - g(Z,X)A(Y) + g(Z,Y)A(X) = 0. \tag{3.20}$$

We know (Mishra - 84) [2]

$$(D_z\,'F)(X,Y) = A(X)g(Y,Z) - A(Y)g(X,Z). \tag{3.21}$$

Barring X and Y

$$(D_z\,'F)(\overline{X},\overline{Y}) = A(\overline{X})g(\overline{Y},Z) - A(\overline{Y})g(\overline{X},Z), \tag{3.22}$$

$$(D_z\,'F)(\overline{X},\overline{Y}) = 0. \tag{3.23}$$

Using (3.8), we get

$$(D_z'F)(\overline{X},\overline{Y}) = A(K(\overline{X},\overline{Y},Z)) = 0. \tag{3.24}$$

From (3.20)

$$\begin{aligned}(D_z\,'F)(X,Y) + A(Y)g(Z,X) - A(X)g(Z,Y) = 0\\ (D_z\,'F)(X,Y) = A(X)g(Z,Y) - A(Y)g(Z,X).\end{aligned} \tag{3.25}$$

Using (1.1)(c), we get

$$(D_z\,'F)(X,Y) = -A(X)g(\overline{Z},\,\overline{Y}) + A(Y)g(\overline{Z},\,\overline{X}). \tag{3.26}$$

Using (3.12), we get

$$(D_z\,'F)(X,Y) = A(Y)(D_z\,'F)(X,T) - A(X)(D_z\,'F)(Y,T). \tag{3.27}$$

Thus we have

Theorem (3.6): On PKCT - Riemannian manifold, we have

$$(D_z\,'F)(X,Y) = A(Y)(D_z\,'F)(X,T) - A(X)(D_z\,'F)(Y,T).$$

Definition (3.2): On PKCT - Riemannian manifold structure $\{F,T,A\}$ is said to be normal if

$$\underset{O}{N}(X,\,Y) = 0, \tag{3.28}$$

where $\underset{O}{N}(X,Y) = \underset{F}{N}(X,Y) + dA(X,Y)T = 0.$ [2]

$$\begin{aligned}\underset{O}{N}(X, Y) =&[\overline{X},\overline{Y}] + [\overline{\overline{X},\overline{Y}}] - [\overline{\overline{X}, Y}] - [\overline{X, \overline{Y}}] + \{XA(Y)\\&- YA(X) - A(X,Y)\}T\\=&D_{\overline{x}}\overline{Y} - D_{\overline{Y}}\overline{X} + [X,Y] - A([X,Y])T - \overline{D_{\overline{X}}}Y + \overline{D_{Y}\overline{\overline{X}}} - \overline{D_{X}\overline{\overline{Y}}}\\&+ \overline{D_{\overline{Y}}X} + \{XA(Y) - YA(X) - A([X,Y])\}T\\=&(D_{\overline{x}}F)(Y) + F(D_{\overline{x}}Y) - (D_{\overline{y}}F)(X) - F(D_{\overline{y}}X) + D_xY - D_yX\\&- A(D_xY)T + A(D_yX)T - \overline{D_{\overline{x}}Y} + (\overline{D_yF)(X)} + \overline{\overline{D_yX}}\\&- \overline{(D_xF)(Y)} - \overline{\overline{D_xY}} + \overline{D_{\overline{y}}X} + (D_xA)(Y)T + A(D_xY)T\\&- (D_yA)(X)T - A(D_yX)T - A(D_xY)T + A(D_yX)T,\\=&(D_{\overline{x}}F)(Y) + \overline{D_{\overline{x}}Y} - (D_{\overline{y}}F)(x) - (\overline{D_{\overline{y}}X}) + D_xY - D_yX\\&- A(D_xY)T + A(D_yX)T - \overline{D_{\overline{x}}Y} + (\overline{D_yF)(X)}\\&+ D_yX - A(D_yX)T - (\overline{D_xF)(Y)} - D_xY + A(D_xY)T + \overline{D_{\overline{y}}X}\\&+ (D_xA)(Y)T + A(D_xY)T - (D_yA)(X)T - A(D_yX)T\\&- A(D_xY)T + A(D_yX)T),\end{aligned}$$

so we have

$$\begin{aligned}\underset{O}{N}(X, Y) =&(D_{\overline{x}}F)(Y) - (D_{\overline{y}}F)(X) + (\overline{D_yF)(X)} - (\overline{D_xF)(Y)}\\&+ \{(D_xA)(Y) - (D_yA)(X)\}T.\end{aligned} \tag{3.29}$$

Differentiating covariantly the equation

$$\overline{\overline{Y}} = F\overline{Y}$$

and using (1.1) and (3.9), we get

$$(\overline{D_xF)(Y)} = -(D_xF)\overline{Y} - (D_xA)(Y)T + A(Y)(\overline{X}). \tag{3.30}$$

Using (3.29) and (3.30), we see that

$$\begin{aligned}\underset{O}{N}(X,Y) = 0 \Leftrightarrow&(D_{\overline{x}}F)(Y) - (D_{\overline{y}}F)(X) - (D_yF)(\overline{X}) - (D_yA)(X)T\\&+ A(X)(\overline{Y}) + (D_xF)(\overline{Y}) + (D_xA)(Y)T - A(Y)\overline{X}\\&+ (D_xA)(Y)T - (D_yA)(X)T = 0,\\\Leftrightarrow&(D_{\overline{x}}F)(Y) - (D_{\overline{y}}F)(X) + (D_xF)(\overline{Y}) - (D_yF)(\overline{X})\\&- A(Y)(\overline{X}) + A(X)(\overline{Y}) + 2((D_xA)(Y) - (D_yA)(X))T = 0.\end{aligned}$$

From (2.1), we get $\underset{O}{N}(X, Y) = 0$, if and only if

$$\begin{aligned}(D_{\overline{x}}F)(Y) - (D_{\overline{y}}F)(X) + (D_xF)(\overline{Y}) - (D_yF)(\overline{X}) - A(Y)(\overline{X}) \\ + A(X)(\overline{Y}) + 4\,'F(X,Y)T = 0,\end{aligned} \tag{3.31}$$

which is equivalent to

$$\begin{aligned}g((D_{\overline{x}}F)Y, Z) - g((D_{\overline{y}}F)X, Z) + g((D_xF)\overline{Y}, Z) - g((D_yF)\overline{X}, Z) \\ - A(Y)g((\overline{X}, Z) + A(X)\,g(\overline{Y}, Z) + 4\,'F(X,Y)\,g(T, Z) = 0,\end{aligned}$$

or

$$\begin{aligned}(D_{\overline{x}}\,'F)(Y,Z) + (D_{\overline{y}}\,'F)(Z, X) + (D_x\,'F)(\overline{Y}, X) - (D_y\,'F)(\overline{X}, Z) \\ - A(Y)g(Z,\overline{X}) + A(X)\,g(Z,\overline{Y}) + 4\,'F(X,Y)\,A(Z) = 0.\end{aligned} \tag{3.32}$$

Using (2.5), (3.32) becomes

$$\begin{aligned}(d\,'F)((\overline{X}, Y, Z) - (D_y\,'F)(Z,\overline{X}) - (D_z\,'F)(\overline{X}, Y) + (d\,'F)(X, \overline{Y}, Z) \\ - (D_x\,'F)(\overline{Y}, Z) - (D_z\,'F)(X,\overline{Y}) + (D_x\,'F)(\overline{Y}, Z) - (D_y\,'F)(\overline{X}, Z) \\ - A(Y)\,'F(X,Z) + A(X)\,'F(Y,Z) + 4\,'F(X,Y)A(Z) = 0\end{aligned}$$

or

$$\begin{aligned}(d\,'F)((\overline{X}, Y, Z) + (d\,'F)(X,\overline{Y}, Z) - (D_z\,'F)(\overline{X}, Y) - (D_z\,'F)(X,\overline{Y})) \\ - A(Y)\,'F(X,Z) + A(X)\,'F(Y,Z) + 4\,'F(X,Y)A(Z) = 0.\end{aligned} \tag{3.33}$$

Since on a APST-Riemannian manifold, we have $(d\,'F) = 0$, the above Equation is equivalent to

$$\begin{aligned}(D_z\,'F)(\overline{X}, Y) + (D_z\,'F)(X,\overline{Y}) = -\ A(Y)\,'F(X,Z) + A(X)\,'F(Y,Z) \\ - 4\,'F(X,Y)\,A(Z).\end{aligned} \tag{3.34}$$

Thus we have

Theorem (3.7): PKCT - Riemannian structure is normal if (3.34) holds.

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