J. T. S.
Vol. 9 (2015), pp.67-75
https://doi.org/10.56424/jts.v9i01.10564

Pseudosymmetric and Holomorphically Pseudosymmetric Manifolds

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(Received: February 2, 2015)

Abstract

We study the Bochner pseudosymmetric , Weyl pseudosymmetric and holomorphic projective pseudosymetric conditions for Khalerian manifold and we show that are not essential in dimension greater than 4. For a 4-dimensional, many interesting relations between the various pseudosymmetry conditions of Kählerian manifolds are determined.

Keywords and Phrases : Kählerian manifold, semi-symmetric, pseudosymmetric Bochner pseudosymmetry, Weyl pseudosymmetry and holomorphic projective pseudosymetry.
2000 AMS Subject Classification : 53C53, 53C25.

1. Introduction

Let M be a Kählerian manifold. Thus, M is a connected $N(=2n)$-dimensional differentiable manifold endowed with an almost Hermitian (g, J) structure, which is formed by a Riemannian metric g and a $(1,1)$-tensor field J (an almost complexe structure) such that

$$J^2 = -Id, \quad g(JX, JY) = g(X, Y), \quad \nabla J = 0$$

for all $X, Y \in \mathfrak{X}(M)$, ∇ is the Levi-Civita connetction corresponding to g and $\mathfrak{X}(M)$ be the Lie algebra of smooth vector fields on M. We denote by $\mathcal{S}$, S and r the Ricci operator, the Ricci curvature $(0,2)$-tensor and the scalar curvature of (M^N, g, J), respectively. $\mathcal{R}(U, V)$ and $U \wedge V$ are define by

$$\mathcal{R}(U, V)Z = [\nabla_U, \nabla_V]Z - \nabla_{[U,V]}Z$$

$$(U \wedge V)Z = g(V,Z)U - g(U,Z)V$$

where $U, V, Z \in \mathfrak{X}(M)$.

The operators $\mathcal{R}(U,V)$ and $U \wedge V$ will be treated as derivations of the tensor algebra on M in the usual sense by assuming that they commute with contraction and vanish on any function on M. The $(0,6)$-tensors fields $R.R$ and the Tachibana tensor $Q(g,R)$ are define by

$$\begin{aligned}(\mathcal{R}.R)(X_1,\dots,X_k;U,V) &= (\mathcal{R}(U,V).R)(X_1,\dots,X_k)\\ &= -R(\mathcal{R}(U,V)X_1,\dots,X_k) - \cdots\\ &\cdots - R(X_1,\dots,\mathcal{R}(U,V)X_k).\end{aligned}$$

$$\begin{aligned}Q(g.R)(X_1,\dots,X_k;U,V) &= \big((U \wedge V).R\big)(X_1,\dots,X_k)\\ &= -R((U \wedge V)X_1,\dots,X_k) - \cdots\\ &\cdots - R(X_1,\dots,(U \wedge V)X_k).\end{aligned}$$

A Kählerian manifold (M,g) is said to be semisymmetric if $\mathcal{R}(U,V).R = 0$.

A Kählerian manifold (M,g) is said to be pseudosymmetric, in the sense of R.Deszcz, if $\mathcal{R}(U,V).R = f\,Q(g,R)$ with f is a real function on the set $U_R = \{x \in M \mid R - \frac{r}{n(n-1)}G \neq 0 \text{ at } x\}$ where G is a $(0,4)$-tensor define by $G(X,Y,Z,W) = g((X \wedge Y)Z,W)$.

It is clear that any semisymmetric Kählerian manifold is pseudosymmetric (for $f = 0$).

F. Defever, R. Deszcz and L. Verstraelen proved (on 1997) that in the class of Kählerian manifolds, the usual pseudosymmetry conditions are not essential(dimension great than four). For dimension 4, Olszak (on 2003) gave an example of non semisymmetric pseudosymmetric Kählerian manifold. In ([1]) and ([10]) is proved that for a 4-dimensional pseudosymmetric non semisymmetric Kählerian manifold M, the Ricci tensor vanishes on the set $\{p \in M, f(p) \neq 0\}$.

We will used freqquently the following proposition.

Proposition 1.1. Let T be $(0,4)$-tensor, on a Kählerian manifold M, satisfying

$$\forall X,Y,Z,W \in \mathfrak{X}(M) \qquad T(X,Y,Z,W) = -T(Y,X,Z,W), \tag{1.1}$$

and

$$\forall p \in M, \forall \{e_i\}_{1 \leq i \leq N} \text{ orthonormal basis of } T_pM, \forall x,y \in T_pM \tag{1.2}$$

$$\sum_{i=1}^{N} T(e_i, x, y, e_i) = 0,$$

In any point $p \in M$ such that $(R.T)_p = f(p)(Q(g,T))_p$ and $f(p) \neq 0$, we have $\forall x, y, z, w, u, v \in T_pM$ $\forall \{e_i\}_{1 \leq i \leq N}$ orthonormal basis of T_pM,

$$\begin{aligned}(3-N)T(u,y,z,w) &= T(z,y,u,w) + T(w,y,z,u) + T(Jy,Ju,z,w) \\ &\quad + T(Jz,y,Ju,w) + T(Jw,y,z,Ju) \\ &\quad + \sum_{i=1}^{N} \Big(g(u,z)T(y,e_i,e_i,w) + g(Ju,y)T(e_i,Je_i,z,w) \\ &\quad - g(Ju,z)T(y,e_i,Je_i,w) + g(Ju,w)T(e_i,y,z,Je_i)\Big)\end{aligned} \tag{1.3}$$

Proof. Knowing that $R(Ju, Jv) = R(u, v)$ and from $(R.T)_p = f(p)(Q(g,T))_p$, we have

$$\begin{aligned}Q(g,T)(x,y,z,w;Ju,Jv) &= \frac{1}{f(p)}\big(R(Ju,Jv).T\big)(x,y,z,w) \\ &= \frac{1}{f(p)}\big(R(u,v).T\big)(x,y,z,w) \\ &= Q(g,T)(x,y,z,w;u,v)\end{aligned} \tag{1.4}$$

so

$$\sum_{i=1}^{N} Q(g,T)(e_i,y,z,w;u,e_i) = \sum_{i=1}^{N} Q(g,T)(e_i,y,z,w;Ju,Je_i) \tag{1.5}$$

by using (1.1) and (1.2), in (1.5) we obtain

$$\begin{aligned}(3-N)T(u,y,z,w) &= T(z,y,u,w) + T(w,y,z,u) + T(Jy,Ju,z,w) \\ &\quad + T(Jz,y,Ju,w) + T(Jw,y,z,Ju) \\ &\quad + \sum_{i=1}^{N} \Big(g(u,z)T(y,e_i,e_i,w) + g(Ju,y)T(e_i,Je_i,z,w) \\ &\quad - g(Ju,z)T(y,e_i,Je_i,w) + g(Ju,w)T(e_i,y,z,Je_i)\Big)\end{aligned}$$

2. Bochner Tensor on Kählerian Manifold

Let (M^N, g, J) a Kählerian manifold of dimension $N = 2n$. The holomorphic Bochner curvature operator $\mathcal{B}$ is defined for all $X, Y \in \mathfrak{X}(M)$ by ([3],[7])

$$\begin{aligned}\mathcal{B}(X,Y) &= \mathcal{R}(X,Y) - \frac{1}{N+4}\Big(X \wedge_g (\mathcal{S}Y) + (\mathcal{S}X) \wedge_g Y + (JX) \wedge_g (\mathcal{S}JY) \\ &+ (\mathcal{S}JX) \wedge_g (JY) - 2g(JX,Y)\mathcal{S}J - 2g(\mathcal{S}JX,Y)J\Big) \\ &+ \frac{r}{(N+4)(N+2)}\Big(X \wedge_g Y + (JX) \wedge_g (JY) - 2g(JX,Y)J\Big).\end{aligned}$$

Recall that the Bochner curvature $(0,4)$-tensor,$B(X,Y,Z,W) = g(\mathcal{B}(X,Y)$ $Z,W)$, has the same algebraic properties as the usual curvature tensor. Moreover, for this tensor, we have

$$\mathcal{B}(X,Y) = -\mathcal{B}(Y,X) \quad , \quad \mathcal{B}(JX,JY) = \mathcal{B}(X,Y) \quad , \quad \mathcal{B}(X,Y)J = J\mathcal{B}(X,Y)$$

$$\mathcal{B}(JX,Y) + \mathcal{B}(X,JY) = 0 \quad , \quad B(X,Y,Z,W) = B(Z,W,X,Y)$$

$$trace\{Z \to \mathcal{B}(Z,X)Y\} = trace\{Z \to \mathcal{B}(JZ,X)Y\} = 0$$

2.1. **Bochner pseudosymetric Kählerian manifold.** A Kählerian manifold (M,J,g) is said to be

- Bochner semisymmetric if $\mathcal{R}(U,V).B = 0$.
- Bochner pseudosymmetric if $\mathcal{R}(U,V).B = f\,(U \wedge V).B$

with f is a real function defined uniquely at every point at which $\mathcal{R}.B \neq 0$
It is clear that any Bochner semisymmetric Kählerian manifold is Bochner pseudosymmetric (for $f = 0$). In other hand, every pseudosymmetric Kählerian manifold is Bochner pseudosymmetric, the converse is not true in general.

Theorem 2.1. Every Bochner pseudosymmetric Kählerian manifold (M,g,J), $dimM > 4$, is Bochner semisymmetric.

Proof. Suppose that M is a Kählerian manifold of dimension $N = 2n$ satisfying $R.B = fQ(g,B)$ with $f \in C^{\infty}(M)$.

Suppose that p is a point in M for which $(R.B)_p \neq 0$. We will derive a contradiction. It is clear that $f(p) \neq 0$. We can applying the proposition (1.1) to B, so from (1.3) and the properties of the tensor B, we obtain

$$(N-4)B(u,y,z,w) = 0$$

and since $N > 4$ then, $B_p = 0$. It is easy to see that $(R.B)_p = 0$, which contradicts our initial assumption. This proves that $R.B = 0$

Remark 2.1. The 4-dimensional case was treated by Z.Olszak in ([10]).

3. Weyl Tensor on Kählerian Manifold

Let (M^N, g, J) a Kählerian manifold of dimension $N = 2n$. The Weyl coformal curvature operator $\mathcal{C}$ is defined for all $X, Y \in \mathfrak{X}(M)$ by ([9])

$$\begin{aligned}\mathcal{C}(X,Y) &= \mathcal{R}(X,Y) - \frac{1}{N-2}\Big(X \wedge_g (\mathcal{S}Y) + (\mathcal{S}X) \wedge_g Y\Big)\\ &+ \frac{r}{(N-1)(N-2)}\Big(X \wedge_g Y\Big).\end{aligned}$$

Notice that this tensor has the following properties

$$\mathcal{C}(X,Y) = -\mathcal{C}(Y,X), \quad C(X,Y,Z,W) = C(Z,W,X,Y),$$

$$C(JX,JY,Z,W) = C(X,Y,JZ,JW),$$

$$tr\{Z \mapsto \mathcal{C}(Z,X)Y\} = 0, \quad \sum_i C(X,e_i,e_i,W)\} = 0$$

3.1. **Weyl pseudosymetric Kählerian manifold.** A Kählerian manifold (M, J, g) is said to be

- Weyl semisymmetric if $\mathcal{R}(U,V).C = 0$.
- Weyl pseudosymmetric if $\mathcal{R}(U,V).C = f\,(U \wedge V).C$

with f is a real function defined uniquely at every point at which $\mathcal{R}.C \neq 0$.

It is clear that any Weyl semisymmetric Kählerian manifold is Weyl pseudosymmetric (for $f = 0$). In other hand, every pseudosymmetric Kählerian manifold is Weyl pseudosymmetric, the converse is not true in general.

Theorem 3.1. Let (M, g, J) be a Weyl pseudosymmetric Kählriann manifold.

- (a): If $\dim M = 4$, M non Weyl semisymmetric and S $\neq 0$, then $r = 0$ and M is not pseudosymmetric.
- (b): If $\dim M = 4$, M non Weyl semisymmetric and S $= 0$, then M is pseudosymmetric.
- (c): If $\dim M > 4$, then M is Weyl semisymmetric.

Proof. Suppose that M is a Kählerian manifold of dimension $N = 2n$ satisfying $R.C = fQ(g,C)$ with $f \in C^\infty(M)$.
Suppose that p is a point in M for which $(R.C)_p \neq 0$ this implies $f(p) \neq 0$.

By applying the proposition (1.1) to C, and using the properties of the tensor C we find,

$$\begin{aligned}(3-N)C(u,y,z,w) &= C(z,y,u,w)+C(w,y,z,u)+C(Jy,Ju,z,w)\\ &+C(Jz,y,Ju,w)+C(Jw,y,z,Ju)\\ &+\sum_{i=1}^{N}\Big(g(Ju,y)C(e_i,Je_i,z,w)-g(Ju,z)C(y,e_i,Je_i,w)\\ &+g(Ju,w)C(e_i,y,z,Je_i)\Big).\end{aligned}$$

contracting y and u after having replaced y by Jy and using again the properties of the tensor C we obtain,

$$\begin{aligned}\sum_{i=1}^{N}\big(2(2-N)C(e_i,Je_i,z,w)\big) &= \sum_{i=1}^{N}\Big(C(z,Je_i,e_i,w)+C(w,Je_i,z,e_i)\\ &-C(z,e_i,Je_i,w)+C(e_i,w,z,Je_i)\Big).\end{aligned}$$

knowing that $C(x,y,z,w)=g(\mathcal{C}(x,y)z,w)$ we can check that

$$(N-4)S(z,Jw)+\frac{r}{N-1}g(z,Jw)=0.$$

- If $N=4$ and $S\neq 0$,then $r=0$ and M is not pseudosymmetric.
- If $N=4$ and $S=0$, then $C=R$ and M is pseudosymmetric.
- If $N>4$, then $S(z,Jw)=-\frac{r}{(N-1)(N-4)}g(z,Jw)$. This implies $R.S=0$. Using this fact in the expression of the tensor C, we obtain $R.C=0$ which contradict $(R.C)_p\neq 0$. So we can conclude that M is Weyl semisymmetric.

Remark 3.1. In this theorem, the result in the case (a) was gotten independently by Z. Olszak ([10]).

4. Holomorphic Projective Tensor on Kählerian Manifold

Let (M^N,g,J) a Kählerian manifold of dimension $N=2n$. The holomorphic projective operator $\mathcal{P}$ is defined for all $X,Y\in\mathfrak{X}(M)$ by ([8])

$$\mathcal{P}(X,Y)=\mathcal{R}(X,Y)-\frac{1}{N+2}\Big(X\wedge_S Y+JX\wedge_S JY-2S(JX,Y)J\Big)$$

Notice that this tensor has the following properties

$$\mathcal{P}(X,Y)=-\mathcal{P}(Y,X),\quad \mathcal{P}(JX,JY)=\mathcal{P}(X,Y),\qquad tr\{Z\mapsto\mathcal{P}(Z,X)Y\}=0,$$

$$\sum_i P(X, e_i, e_i, W)\} = \frac{1}{N+2}\big(NS(X,Y) - rg(X,Y)\big) = \frac{1}{N+2}S_0(X,Y).$$

4.1. **Holomorphic projective pseudosymetric Kählerian manifold.** A Kählerian manifold (M, J, g) is said to be

- holomorphic projective semisymmetric if $\mathcal{R}(U, V).P = 0$.
- holomorphic projective pseudosymmetric if $\mathcal{R}(U, V).P = f\,(U \wedge V).P$

with f is a real function defined uniquely at every point at which $\mathcal{R}.P \neq 0$
It is clear that any holomorphic projective semisymmetric Kählerian manifold is holomorphic projective . In other hand, every pseudosymmetric Kählerian manifold is holomorphic projective pseudosymmetric, the converse is not true in general.

Theorem 4.1. Let (M, g, J) be a holomorphic projective pseudosymmetric Kählerian manifold.

- (a): If $\dim M = 4$, M non holomorphic projective semisymmetric and $r \neq 0$, then $S = \frac{r}{4}g$ and M is not pseudosymmetric .
- (b): If $\dim M = 4$, M non holomorphic projective semisymmetric and $r = 0$, then M is pseudosymmetric .
- (c): If $\dim M > 4$, then M is holomorphic projective semisymmetric.

Proof. Suppose that M is a holomorphic projective pseudosymmetric Kählerian manifold of dimension $N = 2n$ i.e. we have $R.P = fQ(g, P)$ with $f \in C^\infty(M)$. Suppose that p is a point in M for which $(R.P)_p \neq 0$ i.e $f(p) \neq 0$.

Using the proposition (1.1) for P and the properties of the tensor P to find,

$$\begin{aligned}(4-N)P(u,y,z,w) =& P(z,y,u,w) + P(w,y,z,u) + P(Jz,y,Ju,w)\\ &+ P(Jw,y,z,Ju) + \frac{1}{N+2}\Big(g(u,z)S_0(y,w) + 2g(Ju,\\ &y)S_0(z,Jw) + g(Ju,z)S_0(y,Jw)\Big).\end{aligned}$$

Now, contracting u and w and using again the properties of the tensor P to obtain,

$$P(z,y,e_i,e_i) + P(Jz,y,Je_i,e_i) + \frac{4}{N+2}S_0(y,z) = 0,$$

knowing that $P(x,y,z,w) = g(\mathcal{P}(x,y)z,w)$ we can check that

$$S_0(y,z) = 0.$$

i.e.,

$$S(y,z)=\frac{r}{N}g(y,z)$$

that is, the manifold is Einstein. The last equality turns the expression of the tensor P into

$$\mathcal{P}(X,Y)=\mathcal{R}(X,Y)-\frac{r}{N(N+2)}\mathcal{R}^{\mathcal{H}}(X,Y),$$

where $R^{\mathcal{H}}(X,Y)=X\wedge_g Y+(JX)\wedge_g(JY)-2g(JX,Y)J$ then

$$\mathcal{R}.P=\mathcal{R}.R$$

since $\mathcal{R}.R^{\mathcal{H}}=0$ in view of the equality $\nabla R^{\mathcal{H}}=0$.
Knowing that for $\dim M>4$, $\mathcal{R}.R=0$, so $R.P=0$ which contradict $(R.P)_p\neq 0$, then M is holomorphic projective semisymmetric.

For dim $M=4$, we have $S=\frac{r}{4}g$ This gives

$$\mathcal{P}(X,Y)=\mathcal{R}(X,Y)-\frac{r}{24}\mathcal{R}^{\mathcal{H}}(X,Y).$$

If $r=0$ then $P=R$ and M is pseudosymmetric.
If $r\neq 0$ we have

$$Q(g.P)=Q(g.R-\frac{r}{24}R^{\mathcal{H}})=Q(g.R)-\frac{r}{24}Q(g.R^{\mathcal{H}}),$$

since $Q(g.\mathcal{R}^{\mathcal{H}})\neq 0$ in view of $Q(g.J)\neq 0$ then $Q(g.P)\neq Q(g.R)$ and M is not pseudosymmetric.

Example 4.1. ([9]) Let (x^α,y^α,z,t) denote the Cartesian coordinates in $\mathbb{R}^{2m+2}$,$m\geq 1$.Latin indices take on values from 1 to $2m+2$. Greek indices will run from 1 to m and $\alpha'=\alpha+m$ for any $\alpha\in\{1,...,m\}$.
Assume that $M=N\times(A,B)\subset\mathbb{R}^{2m+2}$ where N is an open connected subset of R^{2m+1} et (A,B) is an open interval and $B>A>0$. Suppose that $h:(A,B)\to\mathbb{R}$ is a smooth function which non-zero at any $t\in(A,B)$.
Let (e_i) be the frame of vector fields on M defined by
$e_\alpha=\frac{1}{t}\frac{\partial}{\partial x^\alpha}$, $e_{\alpha'}=\frac{1}{t}(\frac{\partial}{\partial y^\alpha}+2x^\alpha\frac{\partial}{\partial z})$, $e_{2m+1}=\frac{1}{t^2h}\frac{\partial}{\partial z}$, $e_{2m+2}=th\frac{\partial}{\partial t}$
and let (θ^i) be the dual frame of differential 1-forms,
$\theta^\alpha=tdx^\alpha$, $\theta^{\alpha'}=tdy^\alpha$, $\theta^{2m+1}=t^2h(-2\sum_\lambda x^\lambda dy^\lambda+dz)$, $\theta^{2m+2}=\frac{1}{th}dt$
Knowing that the metric g given by $g=\sum\theta^i\otimes\theta^i$ and the almost structure J on M by assuming

$$Je_\alpha=e_{\alpha'}\ ,\ Je_{\alpha'}=-e_\alpha\ ,\ Je_{2m+1}=e_{2m+2}\ ,\ Je_{2m+2}=-e_{2m+1}$$

Thus, (M,g,J) becomes a Kählerian manifold.
For dimension 4 i.e $m=1$ we have,

- For any function h, M is Bochner pseudosymmetric with structure function $f(t) = -2h(h + th')$.
- If $h(t) = \frac{\sqrt{a+bt^2}}{t^3}$ M is Weyl pseudosymmetric non pseudosymmetric with structure function $f(t) = \frac{2(2a+b^2)}{t^6}$ and $r = 0$
- If $h(t) = \frac{\sqrt{a+bt^6}}{t^3}$ M is holomorphic projective pseudosymmetric and non pseudosymmetric with structure function $f(t) = \frac{2(2a-bt^6)}{t^6}$ with $S = 6bg$ and $r = 24b$.
- If $h(t) = \frac{a}{t^3}$ M is Ricci flat and it is pseudosymmetric, Bochner pseudosymmetric, Weyl pseudosymmetric and holomorphic projective pseudosymmetric with the same structure function $f(t) = \frac{4a^2}{t^6}$.

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