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Prolongation of Tensor Fields and G-Structures in Tangent Bundles of Second Order

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Introduction

Tangent and cotangent bundles have been defined and studied by Yano, Ishihara, Patterson and others. Duggal gives the notion of GF-structure, which plays an important role in the differentiable manifold [1]. R. Nivas and Ali have studied the existence of GF-structure and generalized contact structure on the tangent bundle and some interesting results have been obtained for such structures [2]. Prolongation of tensor fields, almost complex and almost product structures have been defined and studied by Yano, Ishihara [3] and others whereas Das [4] and Morimoto [5] have studied the prolongation of F-structure and G-structures respectively to the tangent bundles. In the present paper problems of prolongation in tangent bundle of second order and few results on GF, $f_a(3,-1)$ and generalized contact structures have been discussed.

1. Preliminary

Let M^n be an $n-$dimensional differentiable manifold of class C^∞ and F be a tensor field of type $(1,1)$ satisfying the condition

$$F^2 = a^2 I$$

where $a(\neq 0)$ is any real or complex number, then the structure $\{F\}$ is called $GF-$structure in M^n [1].

If F satisfying the equation

$$F^3 - a^2 F = 0$$

then $\{F\}$ is called $f_a(3,-1)-$structure. If there exists a vector field U and a $1-$form ω in M^n such that

$$F^2 = a^2 I + U \otimes \omega$$

where $FU = 0$, $\omega o F = 0$ and $\omega(U) = -a^2$ then the structure $\{F, U, \omega, a\}$ is called generalized contact structure on M^n.

If $T_p(M^n)$ denotes the tangent space of M^n at $p \in M^n$. Then

$$T(M^n) = \underset{p \in M^n}{\cup} (M^n)$$

is called the tangent bundle of the manifold M^n.

Let M^n be an n- dimensional differentiable manifold and R the real line. We introduce an equivalence relation $\sim$ in the set of all differentiable mapping $F : R \to M^n$. Let $r \geq 1$ be a fixed integer. If two mappings $F : R \to M^n$ and $G : R \to M^n$ satisfy the conditions

$$F^h(0) = G^h(0), \frac{\mathrm{dF^h(0)}}{dt} = \frac{dG^h(0)}{dt}, \ldots\ldots\ldots \frac{d^r F(0)}{dt^r} = \frac{d^r G(0)}{dt^r},$$

the mappings F and G being represented respectively by $x^h = F^h(t)$ and $x^h = G^h(t)$ where $t \in R$ with respect to local coordinates (x^h) in a coordinate neighbourhood (U, x^h) containing the point $P = F(0) = G(0)$, then the mapping F is equivalent to G and written as $F \sim G$. Each equivalence class determined by the equivalence relation $\sim$ is called an $r-$jet of M^n and denoted by $j_p^r(F)$, if this class contains a mapping $F : R \to M^n$ such that $F(0) = P$. The point P is called the target of the $r-$jet $j_p^r(F)$. The set of all $r-$jets of M^n is called the tangent bundle of order r and denoted by $T_r(M^n)$ [3].

2. Prolongation of tensor field and G-structures to the tangent bundle

Let M^n be an $n-$dimensional differentiable manifold and G a Lie subgroup of $GL(n, R)$. A $G-$subbundle $P(M, \pi^*, G)$ of the frame bundle $F(M^n)$ over M^n is called $G-$structure on M^n. That is, a $G-$structure on M^n is a reduction of the structure group $GL(n, R)$ of the tangent bundle $T(M^n)$ to the subgroup G. The tangent bundle $T_2(M^n)$ of order 2 admits a $T_2(G)-$ structure with respect to adapted $3n-$frames $\left\{X_{(i)}^{II}, X_{(i)}^{I}, X_{(i)}^{0}\right\}$ in each π^{-1} (U), $U \in u$, where $\left\{X_{(i)}\right\}$ are $n-$frames adapted to the $G-$structure P in U. The $T_2(G)-$structure thus introduced in $T_2(M^n)$ is called prolongation of the $G-$structure P on M^n to $T_2(M^n)$ and denoted by $\tilde{P}$. The theorem by Yano and Ishihara [3] suggests

"The prolongation $\tilde{P}$ of a $G-$structure P in M^n to $T_2(M^n)$ is integrable if and only if the $G-$structure P is integrable."

Let G be a Lie-subgroup of $GL(n, R)$ and a tensor $\dot{F}$ of type $(1, 1)$ in R^n which is left invariant by G. An $n-$dimensional manifold M^n is assumed to admit a G-structure P. Also the theorem by Yano and Ishihara [3] which states that

"Let T be a tensor field in R^n invariant by a Lie subgroup G of $GL(n, R)$ and P a $G-$structure on an $n-$dimensional manifold M^n. Then, if T is the tensor field induced in M^n from $(\dot{T}, P)$, the tensor field $\tilde{T}$ induced in the tangent bundle $T_2(M^n)$ of order 2 from $(\dot{T}^{II}, \tilde{P})$ is the second lift T^{II} of T to $T_2(M^n)$, where $\dot{T}^{II}$ is the second lift of $\dot{T}$ to $T_2(R^n)$ and $\tilde{P}$ the prolongation of G-structure P to $T_2(M^n)$."

3. Prolongation of G-structures defined by tensor fields to the tangent bundle of second order

In this section, I will discuss three different classical G-structures defined by tensor fields:

(I) $G = GL(n, C)$. Let $\dot{F}$ be a tensor field of type $(1, 1)$ in R^{2n} such that $\dot{F}^2 = a^2 I$ and $G = GL(n, C)$ denote the group of all elements of $GL(2n, R)$ which leave $\dot{F}$ invariant. Then the second lift $\dot{F}^{II}$ of $\dot{F}$ to $T_2(R^{2n})$ is a tensor field of type $(1, 1)$ satisfying $(\dot{F}^{II})^2 = a^2 I$ and the tangent group $T_2(G)$ leaves $\dot{F}^{II}$ invariant. Thus

$$T_2(G) \subset GL(3n, C).$$

Using the above expression along with the earlier mentioned theorems of Yano and Ishihara [3], we have

Theorem 3.1. If M^n admits a GF- structure P determined by a tensor field F of type $(1, 1)$ such that $F^2 = a^2 I$, then on the tangent bundle $T_2(M^n)$ of order 2 the prolongation $\tilde{P}$ of P is a GF-structure which is defined by the second lift F^{II} of F to $T_2(M^n)$. When and only when the GF-structure P is integrable, the prolongation $\tilde{P}$ of P to $T_2(M^n)$ is also integrable.

(II) $G = GL(r, C) \times GL(m, R)$. Let $\dot{F}$ be a tensor of type $(1, 1)$ and of rank r in R^n $(n = 2r + m)$ such that $\dot{F}^3 - a^2\dot{F} = 0$. If we denote by G the group of all elements of $GL(n, R)$, which leave $\dot{F}$ invariant, then we obtain

$$G = GL(r, C) \times GL(m, R) \subset GL(2n, R).$$

Thus the second lift F^{II} of F to $T_2(R^n)$ satisfies

$$(F^{II})^3 - a^2 F^{II} = 0$$

and is of rank $3r$. Hence, we have

$$T_2(G) = GL(3r, C) \times GL(2m, R) \subset GL(3n, R).$$

Using the above relation and again utilizing the theorems of Yano and Ishihara [3], we have

Theorem 3.2. If M^n admits a $f_a(3,-1)-$ structure P defined by a tensor field F of type $(1,1)$ and of rank r everywhere such that $F^3 - a^2F = 0$, then on $T_2(M^n)$ the prolongation $\tilde{P}$ of P admits the similar structure defined by the second lift F^{II} of F to $T_2(M^n)$, where F^{II} is of rank $3r$. When and only when the $f_a(3,-1)-$structure P is integrable in M^n, the prolongation $\tilde{P}$ of P to $T_2(M^n)$ is also integrable.

(III) $G = GL(n, C) \times I$. Let $\dot{F}$ be a tensor field of type (1, 1) and of rank $2n$, $\dot{v}$ a vector field and $\dot{\eta}$ a covector field in R^{2n+1} such that

$$\dot{F}^2 = a^2 I + \dot{v} \otimes \dot{\eta}$$

$$\dot{F}\dot{v} = 0, \qquad \dot{\eta} \circ \dot{F} = 0, \qquad \dot{\eta}(\dot{v}) = -a^2.$$

Thus, if we denote by G the group of all elements of $GL(2n+1, R)$ which leave $\dot{F}$, $\dot{v}$ and $\dot{\eta}$ invariant, then we have

$$G = GL(n, C) \times I \subset GL(2n+1, R)$$

where I denotes the trivial group.

If we put

$$\dot{J} = \dot{f}^{II} + \frac{1}{a}\left\{\dot{v}^0 \otimes \dot{\eta}^0 + \dot{v}^{II} \otimes \dot{\eta}^{II}\right\}$$

$$\dot{U} = \dot{v}^I, \qquad \dot{\omega} = \dot{\eta}^I.$$

We can easily obtain that $(\dot{J}, \dot{U}, \dot{\omega}, a)$ is generalized contact structure in $T_2(R^{2n+1})$. Therefore $T_2(G)$ leaves $\dot{J}$, $\dot{U}$, $\dot{\omega}$ invariant.

Thus we obtain

$$T_2(G) \subset GL(3n+1, C) \times I \subset GL(6n+3, R).$$

Thus we have

Theorem 3.3. If a manifold M^n of dimensions $(2n+1)$ admits generalized almost contact structure P defined by (F, U, ω, a), where $F \in \Im_1^1(M^n)$, $U \in$

$\Im_0^1(M^n)$ and $\omega \in \Im_1^0(M^n)$, then on $T_2(M^n)$ the prolongation $\tilde{P}$ of P is the similar structure defined by $(\tilde{J}, \tilde{U}, \tilde{\omega}, a)$, where

$$\tilde{J} = F^{II} + \frac{1}{a}\left\{U^0 \otimes \omega^0 + U^{II} \otimes \omega^{II}\right\}$$
$$\tilde{U} = U^I, \qquad \tilde{\omega} = \omega^I.$$

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