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A Brief Introduction to Measure Manifold

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Abstract

This talk is about the newly introduced and developed concept of Measure Manifold in terms of measure chart, measure atlas and its various generated categories under different induced invariant structures. The theory of n-dimensional measure manifold and its newly generated categories provide an unified mathematical framework to describe the micro and macro structures.

Keywords : Measure chart, Measure atlas, Measure Manifold.
2010 MSC : 28-XX,49Q15,54-XX,57NXX,58-XX.

1. Introduction

Based on my studies in the field of topology, measure theory, differentiable manifold and topological manifolds, the concept of measure manifold [2] was first introduced in 2014 by S. C. P. Halakatti, by inducing measure structure on differentiable manifold. This newly defined measure manifold carries two structures:
(1) $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $\mathcal{A}_1 \cup \mathcal{A}_2 \in \mathcal{A}^k(M)$,
(2) $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $\mu(\mathcal{A}_1) = k\ \mu(\mathcal{A}_2) \in \mathcal{A}^k(M)$ where $k = 1, 2, ...$

The vision of introducing the concept of measure manifold is to recognise measurable space $(\mathbb{R}^n, \tau, \Sigma)$ as an approximated topological space where topological properties are expressed in terms of the smallest possible open subsets called Borel subsets of $(\mathbb{R}^n, \tau, \Sigma)$. These Borel subsets cover the measurable space $(\mathbb{R}^n, \tau, \Sigma)$ which is closed under countable union and countable intersection. The additional interesting property that measurable space carries is the complement of every Borel subset is a member of $(\mathbb{R}^n, \tau, \Sigma)$. Such a qualified measurable space has a profound intrinsic properties to generate a sound mathematical framework on any measurable manifold (M, τ, Σ) by modeling a Hausdorff second countable space (M, τ) onto a measurable space $(\mathbb{R}^n, \tau, \Sigma)$.

With an appropriate well defined measure function μ defined on $(\mathbb{R}^n, \tau, \Sigma)$ and pulling back measure μ onto a measurable manifold, **the Measure Manifold** (M, τ, Σ, μ) is generated.

The vision is to model Hausdorff second countable topological space (M, τ) onto a measure space $(\mathbb{R}^n, \tau, \Sigma, \mu)$ and define measure manifold in terms of measure charts and measure atlases:

Definition 1.1: Measure chart [2]

A measurable chart $((U, \tau_{/U}, \Sigma_{/U}), \phi)$ is called a measure chart, if $\mu_{/U}$ on $((U, \tau_{/U}, \Sigma_{/U}), \phi)$ satisfies the following conditions:

(i) ϕ is homeomorphism,
(ii) ϕ is measurable function i.e $\phi^{-1}(V) = U \in (U, \tau, \Sigma)$, $V \in (\mathbb{R}^n, \tau_1, \Sigma_1)$ and $(U, \tau_{/U}, \Sigma_{/U}) \subseteq (M, \tau, \Sigma)$,
(iii) ϕ is measure invariant.

Then, the structure $((U, \tau_{/U}, \Sigma_{/U}, \mu_{/U}), \phi)$ is called as a measure chart.

Definition 1.2: Measure atlas [2]

By an $\mathbb{R}^n$-measure atlas of class $C^k (k \geq 1)$ on measure manifold (M, τ, Σ, μ), we mean a countable collection $(\mathcal{A}, \tau_{/\mathcal{A}}, \Sigma_{/\mathcal{A}}, \mu_{/\mathcal{A}})$ of n-dimensional measure charts $((U_i, \tau_{/U_i}, \Sigma_{/U_i}, \mu_{/U_i}), \phi_{i_{/U_i}})$ for all $i \in I$ on (M, τ, Σ, μ) satisfying the following conditions:

($\mathbf{a_1}$) $\cup_{i \in I}(U_i, \tau_{/U_i}, \Sigma_{/U_i}, \mu_{/U_i}) = (M, \tau, \Sigma, \mu)$. That is, the countable union of all measure charts in $(\mathcal{A}, \tau_{/\mathcal{A}}, \Sigma_{/\mathcal{A}}, \mu_{/\mathcal{A}})$ cover (M, τ, Σ, μ_R).

($\mathbf{a_2}$) For any pair of measure charts $((U_i, \tau_{/U_i}, \Sigma_{/U_i}, \mu_{/U_i}), \phi_{i_{/U_i}})$ and $((U_j, \tau_{/U_j}, \Sigma_{/U_j}, \mu_{/U_j}), \phi_{j_{/U_j}})$ in $(\mathcal{A}, \tau_{/\mathcal{A}}, \Sigma_{/\mathcal{A}}, \mu_{/\mathcal{A}})$, the transition maps $\phi_i \circ \phi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ are :

(1) differentiable maps of class C^k $(k \geq 1)$
i.e.,$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j) \subseteq (\mathbb{R}^n, \tau_1, \Sigma_1, \mu_1)$ and $\phi_j \circ \phi_i^{-1}$: $\phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j) \subseteq (\mathbb{R}^n, \tau_1, \Sigma_1, \mu_1)$ are differentiable maps of class C^k $(k \geq 1)$.

(2) measurable, i.e., the transition maps $\phi_i \circ \phi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ are measurable functions if,

(a) any Borel subset $K \subseteq \phi_i(U_i \cap U_j)$ is measurable in $(\mathbb{R}^n, \tau_1, \Sigma_1, \mu_1)$, then $(\phi_i \circ \phi_j^{-1})^{-1}(K) \in \phi_j(U_i \cap U_j)$ is also measurable,

(b) $\phi_j \circ \phi_i^{-1}$ is measurable if $S \subseteq \phi_j(U_i \cap U_j)$ is measurable in $(\mathbb{R}^n, \tau_1, \Sigma_1, \mu_1)$, then $(\phi_j \circ \phi_i^{-1})^{-1}(S) \in \phi_i(U_i \cap U_j)$ is also measurable.

($\mathbf{a_3}$) Any two atlases $(\mathcal{A}_1, \tau_{/\mathcal{A}_1}, \Sigma_{/\mathcal{A}_1}, \mu_{/\mathcal{A}_1})$,$(\mathcal{A}_2, \tau_{/\mathcal{A}_2}, \Sigma_{/\mathcal{A}_2}, \mu_{/\mathcal{A}_2})$ are compatible on (M, τ, Σ, μ) satisfying the two equivalence relations:

i) $\mathcal{A}_1 \sim \mathcal{A}_2$, iff $\mathcal{A}_1 \cup \mathcal{A}_2 \in A^k(M)$,
ii) $\mathcal{A}_1 \sim \mathcal{A}_2$, iff $\mu(\mathcal{A}_1) = k\mu(\mathcal{A}_2) \in A^k(M)$, where $k = 1, 2, ...$

Definition 1.3: Measure Manifold [2]

A measure space (M, τ, Σ, μ) together with a differentiable structure of class C^k and a measure structure induced by μ is called a Measure Manifold of class C^k.

The measure manifold is enriched by inducing the topological properties, say, P that are μ-a.e. on (M, τ, Σ, μ) [10]-[19]. That is, these measurable topological properties hold good only on those Borel subsets, say, A of measure manifold where $\mu(A) > 0$, otherwise, if $\mu(A) = 0$, then the Borel subset A of measure manifold is recognised as **dark region** for that topological property P [10]-[19]. Such topological properties are intrinsic properties of measure manifolds that remain invariant under measurable homeomorphism and measure structure- invariant transformation and enrich the structure of the measure manifold at microscopic level.

By using the above concepts and developed measurable topological properties on the measure manifold, the following different categories of measure manifolds are generated.

I. Different structures induced to generate different categories of measure manifolds:

1. (M, τ, Σ, μ) is a Measure Manifold with the induced σ-algebraic structure and a measure function μ.

Our study on measure manifold with extended topological properties [4][13] has generated the following different categories of measure manifolds:

- Hausdorff measure Manifold.
- Regular measure Manifold.
- Normal measure Manifold.

2. A measure manifold with the induced Atomic measure structure μ_A generates an Atomic measure manifold (M, τ, Σ, μ_A)[7].

Our study on Atomic measure manifold has generated different categories of measure manifolds[7]:

- AT_2 measure Manifold.
- AR measure Manifold.
- AN measure Manifold.

3. A measure manifold with the induced Radon measure structure μ_R generates a Radon measure manifold (M, τ, Σ, μ_R).

Our study on Radon measure manifold with extended topological properties has generated different categories of measure manifolds[8]-[11],[15]-[19]:

- Regular Radon measure Manifold.
- Normal Radon measure Manifold.
- Compact Radon measure Manifold.
- Lindelof Radon measure Manifold.
- Countably compact Radon measure Manifold.
- Semi compact Radon measure Manifold.
- Semi Lindelof Radon measure Manifold.
- Semi countably compact Radon measure Manifold.

4. A measure manifold with the induced path connectedness structure on (M, τ, Σ, μ) generates a Quotient measure manifold $(\mathcal{M}, \tau, \Sigma, \mu)$.

Our study on Quotient Radon measure manifold [15]-[18] has generated the following different categories of path connected measure manifolds and different categories of Quotient Radon measure manifolds:

- Local path connected Radon measure Manifold.
- Internal path connected Radon measure Manifold.
- Maximal path connected Radon measure Manifold.
- Compact Quotient Radon measure Manifold.
- Lindelof Quotient Radon measure Manifold.
- Countably compact Quotient Radon measure Manifold.
- Semi compact Quotient Radon measure Manifold.
- Semi Lindelof Quotient Radon measure Manifold.
- Semi countably compact Quotient Radon measure Manifold.

5. A Quotient measure manifold along with two group structures $(G, \circ)$ and $(\mathcal{G}, \circ)$ on $(\mathcal{M}, \tau, \Sigma, \mu)$ is a Network Measure Manifold $(\mathcal{M}, \tau, \Sigma, \mu_R), (G, \circ), (\mathcal{G}, \circ)$ where $(G, \circ)$ - group of all measurable C^∞ paths of (M, τ, Σ, μ) and $(\mathcal{G}, \circ)$ - group of all measurable homeomorphisms of transition maps.

Our study on Network measure manifolds [15][18] has generated the following different categories of Network measure manifolds:

- Complete Network measure Manifold.
- Compact Network Radon measure Manifold.
- Lindelof Network Radon measure Manifold.
- Countably compact Network Radon measure Manifold.
- Semi compact Network Radon measure Manifold.
- Semi Lindelof Network Radon measure Manifold.
- Semi countably compact Network Radon measure Manifold.

6. A measure manifold with the induced Haar measure structure μ on measurable Fibre Bundle generates Haar measure manifold $(\mathcal{M}, \tau, \Sigma, \mu)$, $(G, \circ), (\mathcal{G}, \circ)$.

Our study on Haar measure manifold $(\mathcal{M}, \tau, \Sigma, \mu), (G, \circ), (\mathcal{G}, \circ)$ with different group structures can generate different categories of measurable Fibre Bundles [14]:

- Measurable Fibre Bundles generated by one parameter groups.
- Measurable Fibre Bundles generated by Lie transformation groups.

2. Conclusion

The theory of n-dimensional measure manifold and its newly generated categories provide an unified mathematical framework to describe micro and macro structures of the n-dimensional measure manifolds. Also measure manifolds have rich intrinsic properties other than metric based properties on manifolds like Riemann metric, Kahlerian metric, Sasakian metric, symplectic structure, etc. The Theory of measure manifold and its categories contribute to the new knowledge in the studies of Theory of manifolds.

The micro analysis developed on the above categories of measure manifolds paves us a way to approximate measurable structures to 4-dimensional measurable manifolds which has a rich applications in the field of cosmology, Quantum physics and Brain building programmes through new gene expression.

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