J. T. S.
Vol. 10 (2016), pp.21-33
https://doi.org/10.56424/jts.v10i01.10577

Bayesian Estimation of the Scale Parameter of a Maxwell Distribution under Asymmetric Loss Functions with Censoring

S. P. Singh

Department of Applied Sciences and Humanities
KIPM College of Engineering and Technology, GIDA, Gorakhpur, India
E-mail : satyap211@gmail.com
(Received: August 12, 2016)

Abstract

The Bayes estimator of the scale parameter of the Maxwell distribution using the squared error, linex, precautionary and entropy loss functions under quasi, natural conjugate and uniform prior distributions have been proposed with the help of type II censored sample. An attempt has been made to obtain the risk functions and corresponding Bayes risks of the estimators for comparing relative efficiency of the procedures.

Keywords and Phrases : Squared error, linex, prior distribution, precautionary and entropy loss function, posterior pdf and expectation, risk and Bayes risk functions.

1. Introduction

Let us consider the Maxwell distribution whose probability density function (pdf) is given by

$$f(x;\theta) = \frac{4}{\sqrt{\pi}} \frac{x^2}{\theta^{3/2}} \; e^{-\frac{x^2}{\theta}}; \qquad \theta > 0, \; x > 0 \tag{1.1}$$

where θ is the scale parameter. The Maxwell distribution plays a very important role of a life time model as for increasing θ the distribution becomes flatter and the right tail increases. Tyagi, R. K. and Bhattacharya, S. K. [7] have given a procedure of MVU estimation of Reliability.

Let us suppose that n items are put to life test and terminate the experiment when $r(< n)$ items have failed. If $x_1, ...,x_r$ denote the first r observations having a common density function as given above, then the joint probability density function is given by

$$f(\underline{x}|\theta) = \frac{n!}{(n-r)!}\left(\frac{4}{\sqrt{\pi}}\right)^r \left(\frac{1}{\theta}\right)^{3r/2} \left(\prod_{i=1}^{r} x_i^2\right) e^{-\left(\frac{T_r}{\theta}\right)} \tag{1.2}$$

where

$$T_r = \left[\sum_{i=1}^{r} x_i^2 + \ (n-r)\ x_{(r)}^2\right].$$

The maximum likelihood estimator (MLE) of θ is given by

$$\hat{\theta} = \frac{2T_r}{3r} \tag{1.3}$$

and the pdf of $\hat{\theta}$ is given by

$$f(\hat{\theta}) = \frac{\left(\frac{3r}{2\theta}\right)^r}{\Gamma\left(\frac{3r}{2}\right)} \left(\hat{\theta}\right)^{\frac{3r}{2}-1} e^{-\frac{3r\,\hat{\theta}}{\theta}}; \qquad \hat{\theta} > 0. \tag{1.4}$$

The fundamental problems in Bayesian analysis are that of the choice of prior distribution $g(\theta)$ and a loss function $L(.,.)$. Let us consider three prior distributions of θ to obtain the Bayes estimators which are given by

(i) Quasi - Prior

$$g_1(\theta) = \frac{1}{\theta^d}; \qquad \theta > 0,\, d > 0, \tag{1.5}$$

here $d = 0$ leads to a diffuse prior and $d = 1$, a non-informative prior.

(ii) Natural Conjugate prior of θ

$$g_2(\theta) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)}\,\theta^{-(\alpha+1)} e^{-\beta/\theta} & ;\ \theta\ >\ 0 \quad (\alpha,\, \beta) > 0 \\ 0 & ;\ \text{otherwise} \end{cases}. \tag{1.6}$$

(iii) Uniform prior

$$g_3(\theta) = \begin{cases} \frac{1}{\beta-\alpha} & ;\ 0 < \alpha \le \theta \le \beta \\ 0 & ;\ \text{otherwise} \end{cases}. \tag{1.7}$$

Loss Function

The Bayes estimator $\hat{\theta}_1$ of θ is of course, optimal relative to the loss function chosen. A commonly used loss function is the squared error loss function (SELF)

$$L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2, \tag{1.8}$$

Some other loss function used are

(i) Linex Loss Function

Varian [8] introduced the following convex loss function known is Linex (linear-exponential) loss function

$$L(\Delta) = be^{a\Delta} - c\Delta - b; \qquad a, c \neq 0, b > 0, \tag{1.9}$$

where $\Delta = \hat{\theta} - \theta$. It is clear that $L(0) = 0$ and the minimum occurs when $ab = c$, therefore, $L(\Delta)$ can be written as

$$L(\Delta) = b[\mathrm{e}^{\mathrm{a}\Delta} - a\Delta - 1], \quad a \neq 0,\ b > 0, \tag{1.10}$$

where a and b are the parameters of the loss function may be defined as shape and scale, respectively. This loss function has been considered by Zellner [9], Rojo [6]. Basu and Ebrahimi [1] considered the $L(\Delta)$ as

$$L(\Delta) = b[e^{\mathrm{a}\Delta} - a\Delta - 1], \quad a \neq 0,\ b > 0, \tag{1.11}$$

where

$$\Delta = \frac{\hat{\theta}}{\theta} - 1.$$

The Bayes estimator under linex loss is denoted by $\hat{\theta}_A$ is obtained by solving the following equation

$$E_\pi \left[\frac{1}{\theta} \exp \left(\frac{a\hat{\theta}_A}{\theta} \right) \right] = e^a \mathrm{E}_\pi \left(\frac{1}{\theta} \right). \tag{1.12}$$

(ii) Precautionary Loss Function

A very useful and simple asymmetric precautionary loss function, proposed in Norstrom [5] is given as

$$L(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\hat{\theta}}. \tag{1.13}$$

The Bayes estimator under this precautionary loss is denoted by $hat\theta_P$ and may be obtained by solving the following equation

$$\hat{\theta}_P = \left[E_\pi \left(\theta^2 \right) \right]^{\frac{1}{2}} \tag{1.14}$$

where $E_\pi(\theta^2)$ denotes the posterior expectation of θ^2.

(iii) Entropy Loss Function

In many practical situations, it appears to be more realistic to express the loss in terms of the ratio $\frac{\hat{\theta}}{\theta}$. In this case, Calabria and Pulcini [2] points out that a useful asymmetric loss function is the entropy loss

$$L(\delta) \propto [\delta^p - p\log_e(\delta) - 1]$$

where $\delta = \frac{\hat{\theta}}{\theta}$.

Also, the loss function $L(\delta)$ has been used in Dey et al. [3] and Dey and Liu [4], in the original form having $p = 1$. Thus $L(\delta)$ can be written as

$$L(\delta) = b[\delta - \log_e(\delta) - 1]; \quad b > 0 \tag{1.15}$$

where $\delta = \frac{\hat{\theta}}{\theta}$.

The Bayes estimator under this entropy loss is denoted by $\hat{\theta}_e$ and may be obtained by solving the following equation

$$\hat{\theta}_e = \left[E_\pi\left(\frac{1}{\theta}\right)\right]^{-1}. \tag{1.16}$$

2. Bayes Estimator under $g_1(\theta)$

The posterior pdf of θ under the prior $g_1(\theta)$, may be obtained, using equation (1.2), as

$$f(\theta|\underline{x}) = \frac{T_r^{(3r+2d-2)/2}}{\Gamma(3r+2d-2)/2}\,\theta^{-(3r+2d)/2}\,e^{-T_r/\theta}; \quad \theta > 0,\ r+d > 1. \tag{2.1}$$

The Bayes estimator under squared error loss function is given by

$$\hat{\theta}_S = \frac{2T_r}{(3r+2d-4)}. \tag{2.2}$$

The Bayes estimator under linex loss function is given by

$$\hat{\theta}_A = \left(1 - e^{-\left(\frac{2a}{3r+2d}\right)}\right)\,\frac{T_r}{a}. \tag{2.3}$$

The Bayes estimator under precautionary loss function is given by

$$\hat{\theta}_P = \frac{2T_r}{[(3r+2d-4)\,(3r+2d-6)]^{1/2}}. \tag{2.4}$$

Also, the Bayes estimator under entropy loss function is given by

$$\hat{\theta}_e = \frac{2T_r}{(3r+2d-2)}. \tag{2.5}$$

The Risk Functions

The risk function of the estimators $\hat{\theta}_S$, $\hat{\theta}_A$, $\hat{\theta}_P$ and $\hat{\theta}_e$ relative to squared error loss function are denoted by $R_S(\hat{\theta}_S)$, $R_S(\hat{\theta}_A)$, $R_S(\hat{\theta}_P)$ and $R_S(\hat{\theta}_e)$, respectively and are given by

$$R_S(\hat{\theta}_S) = \theta^2\left[\frac{3r\,(3r+2)}{(3r+2d-2)^2} - \frac{6r}{(3r+2d-4)} + 1\right], \tag{2.6}$$

$$R_S(\hat{\theta}_A) = \theta^2\left[\frac{3r\,(3r+2)}{4a^2}\left(1-e^{-2a/(3r+2d)}\right)^2 - \frac{3r}{a}\left(1-e^{-2a/(3r+2d)}\right) + 1\right], \tag{2.7}$$

$$R_S(\hat{\theta}_P) = \theta^2\left[\frac{3r(3r+2)}{[(3r+2d-4)(3r+2d-6)]} - \frac{6r}{[(3r+2d-4)(3r+2d-6)]^{1/2}} + 1\right], \tag{2.8}$$

$$R_S\left(\hat{\theta}_e\right) = \theta^2\left[\frac{3r(3r+2)}{[(3r+2d-2)]^2} - \frac{6r}{(3r+2d-2)} + 1\right]. \tag{2.9}$$

The risk function of the estimator $\hat{\theta}_S$, $\hat{\theta}_A$, $\hat{\theta}_P$ and $\hat{\theta}_e$ relative to linex loss function are denoted by $R_A(\hat{\theta}_S)$, $R_A(\hat{\theta}_A)$, $R_A(\hat{\theta}_P)$ and $R_A(\hat{\theta}_e)$, respectively, are given by

$$R_A\left(\hat{\theta}_S\right) = b\left[e^{-a}\left(1-\frac{2a}{3r+2d-4}\right)^{-3r/2} - \left(\frac{3ar}{3r+2d-4}\right) + a - 1\right], \tag{2.10}$$

$$R_A\left(\hat{\theta}_A\right) = b\left[e^{-2ad/(3r+2d)} - \frac{3r}{2}\left(1-e^{-2a/(3r+2d)}\right) + a - 1\right], \tag{2.11}$$

$$R_A\left(\hat{\theta}_P\right) = b\left[e^{-a}\left(1-\frac{2a}{[(3r+2d-4)(3r+2d-6]^{1/2}}\right)^{-3r/2} - \frac{3ar}{[(3r+2d-4)(3r+2d-6]^{1/2}} + a - 1\right] \tag{2.12}$$

and

$$R_A\left(\hat{\theta}_e\right) = b\left[e^{-a}\left(1-\frac{2a}{3r+2d-2}\right)^{-3r/2} - \frac{3ar}{3r+2d-2} + a - 1\right]. \tag{2.13}$$

The risk function of the estimator $\hat{\theta}_S$, $\hat{\theta}_A$, $\hat{\theta}_P$ and $\hat{\theta}_e$ relative to precautionary loss function is denoted by $R_P(\hat{\theta}_S)$, $R_P(\hat{\theta}_A)$, $R_P(\hat{\theta}_P)$ and $R_P(\hat{\theta}_e)$, respectively and are written as

$$R_P\left(\hat{\theta}_S\right) = \theta\left[\frac{3r+2d-4}{3r-2} + \frac{3r}{3r+2d-4} - 2\right], \tag{2.14}$$

$$R_P(\hat{\theta}_A) = \theta\left[\frac{2a}{(3r-2)\left(1-e^{-2a/(3r+2d)}\right)} + \frac{3r\left(1-e^{-2a/(3r+2d)}\right)}{2a} - 2\right], \tag{2.15}$$

$$R_P(\hat{\theta}_P) = \theta \left[\frac{[(3r+2d-4)(3r+2d-6)]^{1/2}}{3r-2} + \frac{3r}{[(3r+2d-4)(3r+2d-6)]^{1/2}} - 2 \right], \tag{2.16}$$

and

$$R_P(\hat{\theta}_e) = \theta \left[\frac{3r+2d-2}{(3r-2)} + \frac{3r}{(3r+2d-2)} - 2 \right]. \tag{2.17}$$

The risk function of the estimator $\hat{\theta}_S$, $\hat{\theta}_A$, $\hat{\theta}_P$ and $\hat{\theta}_e$ relative to entropy loss function are denoted by $R_e(\hat{\theta}_S)$, $R_e(\hat{\theta}_A)$, $R_e(\hat{\theta}_P)$ and $R_e(\hat{\theta}_e)$, respectively and are given as

$$R_e(\hat{\theta}_s) = b \left[\frac{2(2-d)}{(3r+2d-4)} - \mathrm{E}_\theta \mathrm{log}_e \left(\frac{\hat{\theta}_s}{\theta} \right) \right], \tag{2.18}$$

$$R_e(\hat{\theta}_A) = b \left[\frac{3r\left(1 - e^{-2a/(3r+2d)}\right)}{2a} - \mathrm{E}_\theta \mathrm{log}_e \left(\frac{\hat{\theta}_A}{\theta} \right) - 1 \right]. \tag{2.19}$$

$$R_e(\hat{\theta}_P) = b \left[\frac{3r}{[(3r+2d-4)(3r+2d-6)]^{1/2}} - \mathrm{E}_\theta \mathrm{log}_e \left(\frac{\hat{\theta}_P}{\theta} \right) - 1 \right]. \tag{2.20}$$

and

$$R_e(\hat{\theta}_e) = b \left[\frac{2(1-d)}{(3r+2d-2)} - \mathrm{E}_\theta \mathrm{log}_e \left(\frac{\hat{\theta}_e}{\theta} \right) \right]. \tag{2.21}$$

3. Bayes Estimator under $g_2(\theta)$

The posterior pdf of θ under $g_2(\theta)$, using equation (1.2) may be obtained as

$$f(\theta|\underline{x}) = \frac{(\beta+T_r)^{(3r+2\alpha)/2}}{\Gamma\left(\frac{3r+2\alpha}{2}\right)} \theta^{-\frac{3r+2\alpha+2}{2}} e^{-(\beta+T_r)/\theta}. \tag{3.1}$$

The Bayes estimator under squared error loss function is given by

$$\hat{\theta}_S = \frac{2(\beta+T_r)}{(3r+2\alpha-2)}. \tag{3.2}$$

The Bayes estimator under linex loss function comes out to be

$$\hat{\theta}_A = \left(\frac{1 - e^{-2a/(3r+2\alpha+2)}}{a} \right) (\beta+T_r). \tag{3.3}$$

The Bayes estimator under precautionary loss function is obtained as

$$\hat{\theta}_P = \frac{2(\beta+T_r)}{[(3r+2\alpha-2)(3r+2\alpha-4)]^{1/2}}. \tag{3.4}$$

The Bayes estimator under entropy loss function may be written as

$$\hat{\theta}_e = \frac{2(\beta + T_r)}{(3r + 2\alpha)}. \tag{3.5}$$

The Risk Functions

The risk function of the estimators $\hat{\theta}_S$, $\hat{\theta}_A$, $\hat{\theta}_P$ and $\hat{\theta}_e$ relative to squared error loss function are denoted by $R_S(\hat{\theta}_S)$, $R_S(\hat{\theta}_A)$, $R_S(\hat{\theta}_P)$ and $R_S(\hat{\theta}_e)$, respectively and are given by

$$R_S(\hat{\theta}_S) = \theta^2 \left[\left(\frac{3r(3r+2) + \frac{12r\beta}{\theta} + \frac{4\beta^2}{\theta^2}}{(3r + 2\alpha - 2)^2} \right) - \frac{2\left(3r + \frac{2\beta}{\theta}\right)}{(3r + 2\alpha - 2)} + 1 \right], \tag{3.6}$$

$$R_S(\hat{\theta}_A) = \theta^2 \left[C^2 \left(\frac{3r}{4}(3r+2) + \frac{3r\beta}{\theta} + \frac{\beta^2}{\theta^2} \right) - C\left(3r + \frac{2\beta}{\theta}\right) + 1 \right], \tag{3.7}$$

where

$$C = \left(\frac{1 - e^{-2a/(3r+2\alpha+2)}}{a} \right).$$

$$R_S(\hat{\theta}_P) = \theta^2 \left[K^2 \left(\frac{3r}{4}(3r+2) + 3r\left(\frac{\beta}{\theta}\right) + \left(\frac{\beta}{\theta}\right)^2 \right) - K\left(3r + \left(\frac{2\beta}{\theta}\right)\right) + 1 \right], \tag{3.8}$$

where $K = 2\left[(3r + 2\alpha - 2)(3r + 2\alpha - 4)\right]^{-1/2}$. Also

$$R_S(\hat{\theta}_e) = \theta^2 \left[\frac{3r(3r+2) + 12r\left(\frac{\beta}{\theta}\right) + 4\left(\frac{\beta}{\theta}\right)^2}{[(3r + 2\alpha)]^2} - \frac{2\left(3r + \frac{2\beta}{\theta}\right)}{(3r + 2\alpha)} + 1 \right]. \tag{3.9}$$

The risk function of the estimator $\hat{\theta}_S$, $\hat{\theta}_A$, $\hat{\theta}_P$ and $\hat{\theta}_e$ relative to linex loss function are denoted by $R_A(\hat{\theta}_S)$, $R_A(\hat{\theta}_A)$, $R_A(\hat{\theta}_P)$ and $R_A(\hat{\theta}_e)$, respectively and are given by

$$R_A(\hat{\theta}_S) = b \left[\left(e^{-a\left(1 - \frac{2\beta}{\theta(3r+2\alpha-2)}\right)} \right) \left(1 - \frac{2a}{3r + 2\alpha - 2} \right)^{-3r/2} - \left(\frac{a\left(3r + \frac{2\beta}{\theta}\right)}{3r + 2\alpha - 2} \right) + a - 1 \right], \tag{3.10}$$

$$R_A(\hat{\theta}_A) = b \left[\left(e^{-2a(\alpha+1)/(3r+2\alpha+2)} \right) \left(e^{\frac{\beta}{\theta}(1 - e^{-2a/(3r+2\alpha+2)})} \right) \right.$$

$$-\frac{1}{2}\left(1-e^{-2a/(3r+2\alpha+2)}\right)\left(3r+\frac{2\beta}{\theta}\right)+a-1\Bigg], \tag{3.11}$$

$$R_A(\hat{\theta}_P)=b\left[(1-aK)^{-3r/2}\,e^{-a\left(1-\frac{\beta K}{\theta}\right)}-aK\left(\frac{3r}{2}+\frac{\beta}{\theta}\right)+a-1\right], \tag{3.12}$$

where $K=2[(3r+2\alpha-2)(3r+2\alpha-4)]^{-1/2}$. Also

$$R_A(\hat{\theta}_e)=b\left[\left(1-\frac{2a}{3r+2\alpha}\right)^{-3r/2}\exp\left\{-a\left(1-\frac{2\beta}{\theta(3r+2\alpha)}\right)\right\}\right.$$

$$\left.-\frac{a\left(3r+\frac{2\beta}{\theta}\right)}{(3r+2\alpha)}+a-1\right]. \tag{3.13}$$

The risk function of the estimator $\hat{\theta}_S$, $\hat{\theta}_A$, $\hat{\theta}_P$ and $\hat{\theta}_e$ relative to entropy loss function are denoted by $R_e(\hat{\theta}_S)$, $R_e(\hat{\theta}_A)$, $R_e(\hat{\theta}_P)$ and $R_e(\hat{\theta}_e)$, respectively are given by

$$R_e(\hat{\theta}_s)=b\left[\frac{\left(3r+\frac{2\beta}{\theta}\right)}{(3r+2\alpha-2)}-\mathrm{E}_\theta\log_e\left(\frac{\hat{\theta}_s}{\theta}\right)-1\right], \tag{3.14}$$

$$R_e(\hat{\theta}_A)=b\left[\frac{1}{a}\left\{\left(1-e^{-2a/(3r+2\alpha+2)}\right)\left(\frac{3r}{2}+\frac{\beta}{\theta}\right)\right\}-\mathrm{E}_\theta\log_e\left(\frac{\hat{\theta}_A}{\theta}\right)-1\right], \tag{3.15}$$

$$R_e(\hat{\theta}_P)=b\left[\left(\frac{3r}{2}+\frac{\beta}{\theta}\right)K-\mathrm{E}_\theta\log_e\left(\frac{\hat{\theta}_P}{\theta}\right)-1\right], \tag{3.16}$$

where $K=2[(3r+2\alpha-2)(3r+2\alpha-4)]^{-1/2}$ and

$$R_e(\hat{\theta}_e)=b\left[\frac{\left(3r+\frac{2\beta}{\theta}\right)}{(3r+2\alpha)}-\mathrm{E}_\theta\log_e\left(\frac{\hat{\theta}_e}{\theta}\right)-1\right]. \tag{3.17}$$

The Bayes risks

The Bayes risks for the estimators $\hat{\theta}_S$, $\hat{\theta}_A$, $\hat{\theta}_P$ and $\hat{\theta}_e$ are the prior expectations of the risk obtained above. These risks are denoted by $r_S(\hat{\theta}_S)$, $r_S(\hat{\theta}_A)$, $r_S(\hat{\theta}_P)$ and $r_S(\hat{\theta}_e)$ under squared error loss function respectively, and are given by

$$r_S(\hat{\theta}_S)=\frac{2\beta^2}{(\alpha-1)(\alpha-2)(3r+2\alpha-2)}, \tag{3.18}$$

$$r_S(\hat{\theta}_A)=\beta^2\left[\frac{3r(3r+2)C^2-12rC+4}{4(\alpha-1)(\alpha-2)}+\frac{C(3rC-2)}{(\alpha-1)}+C^2\right], \tag{3.19}$$

$$r_S(\hat{\theta}_P) = \beta^2 \left[\frac{3r(3r+2)K^2 - 12rK + 4}{4(\alpha-1)(\alpha-2)} + \frac{K(3rK-2)}{(\alpha-1)} + K^2\right], \tag{3.20}$$

and

$$r_S(\hat{\theta}_e) = \beta^2 \left[\frac{3r(3r+2)\, C_1^2 - 12rC_1 + 4}{4(\alpha-1)(\alpha-2)} + \frac{C_1(3rC_1-2)}{(\alpha-1)} + C_1^2\right], \tag{3.21}$$

where $C_1 = \frac{2}{(3r+2\alpha)}$.

The Bayes risks for the estimators $\hat{\theta}_S$, $\hat{\theta}_A$, $\hat{\theta}_P$ and $\hat{\theta}_e$ relative to linex loss function are denoted by $r_A(\hat{\theta}_S)$, $r_A(\hat{\theta}_A)$, $r_A(\hat{\theta}_P)$ and $r_A(\hat{\theta}_e)$, respectively, and are given by

$$r_A(\hat{\theta}_S) = b\left[e^{-a}\left(1 - \frac{2a}{3r+2\alpha-2}\right)^{-(3r+2\alpha)/2} - \left(1 + \frac{2a}{3r+2\alpha-2}\right)\right], \tag{3.22}$$

$$r_A(\hat{\theta}_A) = b\left[a - \frac{1}{2}(3r+2\alpha+2)\left(1 - e^{-2a/(3r+2\alpha+2)}\right)\right], \tag{3.23}$$

$$r_A(\hat{\theta}_P) = b\left[e^{-a}(1-aK)^{-(3r+2\alpha)/2} - \frac{aK}{2}(3r+2\alpha) + a - 1\right], \tag{3.24}$$

$$r_A(\hat{\theta}_e) = b\left[e^{-a}\left(1 - \frac{2a}{3r+2\alpha}\right)^{-(3r+2\alpha)/2} - 1\right]. \tag{3.25}$$

The Bayes risks for the estimators $\hat{\theta}_S$, $\hat{\theta}_A$, $\hat{\theta}_P$ and $\hat{\theta}_e$ relative to entropy loss function are denoted by $r_e(\hat{\theta}_S)$, $r_e(\hat{\theta}_A)$, $r_e(\hat{\theta}_P)$ and $r_e(\hat{\theta}_e)$, respectively, and are given by

$$r_e(\hat{\theta}_S) = b\left[\frac{2}{(3r+2\alpha-2)} - E\left\{E_\theta\left(\log_e \frac{\hat{\theta}_S}{\theta}\right)\right\}\right], \tag{3.26}$$

$$r_e(\hat{\theta}_A) = b\left[\left(\frac{3r}{2}+a\right)\left(\frac{1-e^{-2a/(3r+2\alpha+2)}}{a}\right) - E\left\{E_\theta\left(\log_e \frac{\hat{\theta}_A}{\theta}\right)\right\} - 1\right], \tag{3.27}$$

$$r_e(\hat{\theta}_P) = b\left[K(3r+2\alpha) - E\left\{E_\theta\left(\log_e \frac{\hat{\theta}_P}{\theta}\right)\right\} - 1\right], \tag{3.28}$$

and

$$r_e(\hat{\theta}_e) = -b\left[E\left\{E_\theta\left(\log_e \frac{\hat{\theta}_e}{\theta}\right)\right\}\right]. \tag{3.29}$$

4. Bayes Estimator under $g_3(\theta)$

The posterior pdf of θ under $g_3(\theta)$, may be obtained as

$$f(\theta|\underline{x}) = \frac{T_r^{(3r-2)/2}\, \theta^{-3r/2}\, e^{-T_r/\theta}}{I_g\left(\frac{T_r}{\alpha},\, \frac{3r}{2}-1\right) \;-\; I_g\left(\frac{T_r}{\beta},\, \frac{3r}{2}-1\right)}, \tag{4.1}$$

where $I_g\,(x,n) = \int\limits_0^x e^{-t}\, t^{n-1}\, dt$ is the incomplete gamma function. The Bayes estimator of θ under squared error loss function is given by

$$\hat{\theta}_S = \left(\frac{I_g\left(\frac{T_r}{\alpha},\, \frac{3r}{2}-2\right) - I_g\left(\frac{T_r}{\beta},\, \frac{3r}{2}-2\right)}{I_g\left(\frac{T_r}{\alpha},\, \frac{3r}{2}-1\right) - I_g\left(\frac{T_r}{\beta},\, \frac{3r}{2}-1\right)} \right) T_r. \tag{4.2}$$

The Bayes estimator of θ under linex loss function given by

$$e^a \frac{I_g\left(\frac{T_r}{\alpha},\, \frac{3r}{2}\right) - I_g\left(\frac{T_r}{\beta},\, \frac{3r}{2}\right)}{I_g\left(\frac{T_r - a\hat{\theta}_A}{\alpha},\, \frac{3r}{2}\right) - I_g\left(\frac{T_r - a\hat{\theta}_A}{\beta},\, \frac{3r}{2}\right)} = \left(\frac{T_r}{T_r - a\hat{\theta}_A} \right)^{3r/2}. \tag{4.3}$$

The Bayes estimator of θ under precautionary loss function is given by

$$\hat{\theta}_P = \left[\frac{I_g\left(\frac{T_r}{\alpha},\, \frac{3r}{2}-3\right) - I_g\left(\frac{T_r}{\beta},\, \frac{3r}{2}-3\right)}{I_g\left(\frac{T_r}{\alpha},\, \frac{3r}{2}-1\right) - I_g\left(\frac{T_r}{\beta},\, \frac{3r}{2}-1\right)} \right]^{1/2} T_r. \tag{4.4}$$

The Bayes estimator of θ under entropy loss function is given by

$$\hat{\theta}_e = \left[\frac{I_g\left(\frac{T_r}{\alpha},\, \frac{3r}{2}-1\right) - I_g\left(\frac{T_r}{\beta},\, \frac{3r}{2}-1\right)}{I_g\left(\frac{T_r}{\alpha},\, \frac{3r}{2}\right) - I_g\left(\frac{T_r}{\beta},\, \frac{3r}{2}\right)} \right] T_r. \tag{4.5}$$

In this case risk functions and Bayes risks cannot be obtained in a closed form.

The Comparisons

It is evident from the equations (1.3), (2.2), (2.3), (2.4), (2.5), (3.2), (3.3), (3.4), (3.5), (4.2), (4.3), (4.4) and (4.5) that the MLE $\hat{\theta}$ Bayes estimators of the scale parameter of the Maxwell distribution, under squared error, linex, precautionary and entropy loss functions using quasi, natural conjugate and uniform priors, have different expressions for their definitions. The Bayes estimators do depend upon the parameters of the prior distributions.

In figure-1, we have plotted the risk functions B_1, B_2, B_3 and B_4 of the Bayes estimators $\hat{\theta}_S$, $\hat{\theta}_A$, $\hat{\theta}_P$ and $\hat{\theta}_e$, respectively, under squared error loss function, as given in equation (2.6), (2.7), (2.8) and (2.9) for $a = 1$, $r = 6(6)24$ and $d = 0.5(0.5)5.0$.

Fig. 1.

Fig. 2.

In figure-2, we have plotted the risk functions C_1, C_2, C_3 and C_4 of the Bayes estimators $\hat{\theta}_S$, $\hat{\theta}_A$, $\hat{\theta}_P$ and $\hat{\theta}_e$, respectively, under linex loss function, as given in equation (3.6), (3.7), (3.8) and (3.9) for $a = 1$, $r = 6(6)24$ and $d = 0.5(0.5)5.0$.

In figure-3, we have plotted the risk functions D_1, D_2, D_3 and D_4 of the Bayes estimators $\hat{\theta}_S$, $\hat{\theta}_A$, $\hat{\theta}_P$ and $\hat{\theta}_e$, respectively, under precautionary loss function, as given in equation (2.14), (2.15), (2.16) and (2.17) for $a = 1$, $r = 6(6)24$ and $d = 0.5(0.5)5.0$.

Fig. 3.

From figure-1, 2 and 3 it is clear that neither of the estimators uniformly dominates the other.

References

[1] Basu, A. and Ebrahimi, N. : Bayesian approach to life testing and reliability estimation using asymmetric loss function, J. Statist. Plan. Inf., 29 (1991), 21-31.

[2] Calabria, R. and Pulcini, G. : Point estimation under asymmetric loss function for left truncated esponential samples, Commun. Statist. Theory and Methods, 25 (3) (1996), 585-600.

[3] Dey, D. K., Ghosh, M. and Srinivasan, C. : Simultaneous estimation of parameters under entropy loss, Jour. Statist. Plan. and Infer., (1987), 347-363.

[4] Dey, D. K. and Pei-San Liao Liu : On comparison of estimators in a generalized life model, Microelectron. Reliab., 32 (1/2), 207-221.

[5] Norstrom, J. G. : The use of precautionary loss functions in risk analysis, IEEE Trans. Reliab., 45 (3) (1996), 400-403.

[6] Rojo, J. : On admissibility of $cx + d$ with respect to the linex loss function, Common. Statist. Theory Methods, 16 (1987), 3745-3748.

[7] Tyagi, R. K. and Bhattacharya, S. K. : A note on the MVU estimation of reliability for the Maxwell failure distribution, Estadistica, 41 (1989), 73-79.

[8] Varian, H. R. : A Bayesian approach to real state assessment in studies in Bayesian Econometrics and Statistics in honour of L. J. Savage, Eds. S. E. Feinberg and A. Zellner, Amsterdam, North Holland, (1975), 195-208.

[9] Zellner, A. : Bayes estimation and prediction using asymmetric loss functions, Jour. Amer. Statist. Assoc., 81 (1986), 446-451.