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A Study of CR-Structures and $F_\lambda(2\nu+3,4)-$HSU-Structures Satisfying $F^{2\nu+3}+\lambda^r F^4=0$

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Abstract

$CR-$structure and F structures satisfying certain equations have been discussed by different authors ([1], [2], [3], [4], [5], [6], [8], [9]). In this paper, we consider $CR-$structures and $F_\lambda(2\nu+3,4)-$Hsu-structure satisfying $F^{2\nu+3}+\lambda^r F^4=0$ and obtain certain results. Also, we discuss the integrability condition for $F_\lambda(2\nu+3,4)-$Hsu structure.

1. Introduction

Let M be an $n-$dimensional differentiable manifold of class C^∞. Suppose there exists on M a non-zero tensor field F of type (1, 1) and of class C^∞ satisfying

$$F^{2\nu+3}+\lambda^r F^4=0, \tag{1.1}$$

where ν is a fixed positive integer greater than or equal to 1, and r is any positive integer, λ a non-zero complex number. The rank of F is constant everywhere and equal to r.

Let us define the operators on M as follows

$$l=-\frac{F^{2\nu+1}}{\lambda^r} \qquad \text{and} \qquad m=I+\frac{F^{2\nu+1}}{\lambda^r}, \tag{1.2}$$

where I denotes the identity operator on M, then it is easy to show that

$$l^2=l, \qquad m^2=m, \qquad l+m=I, \qquad lm=ml=0. \tag{1.3}$$

Hence the operators l and m, when applied to the tangent space of M are complementary projection operators. Let us call such a structure on M as $F_\lambda(2\nu+3,4)-$Hsu structure [9] of rank r.

If D_l and D_m are complementary distributions corresponding to operators l and m respectively, then we can easily show that $\frac{F^{\nu-\frac{1}{2}}}{\lambda^{r/2}}$ acts on D_l as an almost complex structure operator and D_m as a null operator.

The Nijenhuis tensor $N(X,Y)$ of F satisfying equation (1.1) in M is given by

$$N(X,Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y]. \tag{1.4}$$

A necessary and sufficient condition for the $F_\lambda(2\nu+3,4)-$Hsu structure to be integrable is that, $N(X,Y) = 0$ for any two vector fields X and Y on M. For the vector fields X and Y on M, their Lie bracket $[X,Y]$ is defined by

$$[X,Y] = XY - YX. \tag{1.5}$$

2. CR-structure

Let M be a differtiable manifold and $T_C(M)$ be its complexified tangent bundle. A $CR-$Structure on M is a complex sub-bundle H of $T_C(M)$ such that $H_P \cap \tilde{H}_P = 0$ and H is involutive, that is for complex vector fields X and Y in H, $[X,Y]$ is in H. In this case M is a $CR-$Submanifold. Here $\tilde{H}_P$ is complex conjugate of H_P.

Let $F_\lambda(2\nu+3,4)-$Hsu structure be integrable structure satisfying equation (1.1) of rank $r = 2m$ on M. We define complex subbundle H of $T_C(M)$ by

$$H_P = \{X - \sqrt{-1}FX;\ X \in \chi(D_l)\}, \tag{2.1}$$

where $\chi(D_l)$ is the $F(D_m)$ module of all differentiable sections of D_l, then

$$R_e(H) = D_l \qquad \text{and} \qquad H_P \cap \tilde{H}_P = 0.$$

Theorem 2.1. If P and Q are two elements of H, then the following relation holds

$$[P,Q] = [X,Y] - [FX,FY] - \sqrt{(-1)}(-1)([X,FY] + [FX,Y]). \tag{2.2}$$

Proof. Let us define

$$P = X - \sqrt{(-1)}(-1)FX \quad \text{and} \quad Q = Y - \sqrt{(-1)}(-1)FY.$$

Then by direct calculation and simplification, we get

$$\begin{aligned}[P,Q] &= [X-\sqrt{(-1)}(-1)FX, Y-\sqrt{(-1)}(-1)FY] \\ &= [X,Y]-\sqrt{(-1)}(-1)[X,FY]-\sqrt{(-1)}(-1)[FX,Y]-[FX,FY] \\ &= [X,Y]-[FX,FY]-\sqrt{(-1)}(-1)([X,FY]+[FX,Y]).\end{aligned}$$

Theorem 2.2. If $F_\lambda(2\nu+3,4)-$Hsu structure satisfying equation (1.1) is integrable, then we have

$$-\frac{F^{2\nu-2}}{\lambda^r}(F[FX,FY]+F^2[X,Y]) = l([FX,Y]+[X,FY]). \tag{2.3}$$

Proof. Since $N(X,Y)=0$, then from equation (1.4), we have

$$[FX,FY]+F^2[X,Y] = F[X,FY]+F[FX,Y]. \tag{2.4}$$

Operating by $-\frac{F^{2\nu-2}}{\lambda^r}$, we get

$$\begin{aligned}-\frac{F^{2\nu-2}}{\lambda^r}([FX,FY]+F^2[X,Y]) &= -\frac{F^{2\nu-2}}{\lambda^r}(F[X,FY]+[FX,Y]) \\ &= -\frac{F^{2\nu-1}}{\lambda^r}([FX,Y]+[X,FY].\end{aligned} \tag{2.5}$$

Making use of equation (1.2), above equation (2.5) takes the form

$$-\frac{F^{2\nu-2}}{\lambda^r}(F[FX,FY]+F^2[X,Y]) = l([FX,Y]+[X,FY])$$

which proves the theorem.

Theorem 2.3. The following identities hold in a $F_\lambda(2\nu+3,4)-$Hsu structure manifold M

$$mN(X,Y)=0, \tag{2.6}$$

$$mN\left(\frac{F^{2\nu-2}}{\lambda^r}X, Y\right) = m\left[\frac{F^{2\nu-1}}{\lambda^r}X, FY\right]. \tag{2.7}$$

Proof. The proof of above theorem is straight forward.

Theorem 2.4. In a $(F_\lambda(2\nu+3,4)-$Hsu structure manifold M, for any two vector fields X and Y, the following conditions are equivalent:

$$\begin{array}{ll} \text{(i)} & mN(X,Y)=0,\\ \text{(ii)} & m[FX,FY]=0,\\ \text{(iii)} & mN\left(\dfrac{F^{2\nu}}{\lambda^r}X,FY\right)=0,\\ \text{(iv)} & mN\left(\dfrac{F^{2\nu-1}}{\lambda^r}X,Y\right)=0,\\ \text{(v)} & mN\left(\dfrac{F^{2\nu-2}}{\lambda^r}lX,FY\right)=0. \end{array} \tag{2.8}$$

Proof. The proof of equation (2.8) follows easily by virtue of equations (1.1), (1.2), (1.4) and (2.3).

Theorem 2.5. If $\frac{F^{2\nu-1}}{\lambda^r}$ acts on D_l as an almost complex structure, then

$$m\left[\frac{F^{2\nu}}{\lambda^r}lX,FY\right]=m[-FX,FY]=0. \tag{2.9}$$

Proof. We have

$$\begin{aligned} m\left[\frac{F^{2\nu}}{\lambda^r}lX,FY\right] &= m\left[\left(\frac{F^{2\nu-1}}{\lambda^r}\right)FlX,FY\right]\\ &= m[-lFlX,FY]=m[-FX,FY]=0. \end{aligned}$$

Theorem 2.6. For any $X,Y\in\chi(D_l)$, we get

$$l([X,FY]+[FX,Y])=[X,FY]+[FX,Y]. \tag{2.10}$$

Proof. Since $[X,FY]$ and $[FX,Y]\in\chi(D_l)$, and $Fl=lF$; $Fm=mF=0$, then making use of equation (1.5), we have

$$\begin{aligned} l([X,FY]+[FX,Y]) &= l\{X.FY-FY.X+FX.Y-Y.FX\}\\ &= X.FY-FY.X+FX.Y-Y.FX\\ &= [X,FY]+[FX,Y]. \end{aligned}$$

Theorem 2.7. The integrable $F_\lambda(2\nu+3,4)-$Hsu structure satisfying equation (1.1) on M defines a $CR-$structure H on it such that $R_eH=D_l$.

Proof. In view of the fact that $[X, FY]$ and $[FX, Y] \in \chi(D_l)$, and on using equations (2.2), (2.4) and theorem (2.6), we get

$$\begin{aligned} l[P,Q] &= l[X,Y] - l[FX,FY] - \sqrt{(-1)}(-1)l([X,FY] + [FX,Y]) \\ &= [X,Y] - [FX,FY] - \sqrt{(-1)}(-1)([X,FY] + [FX,Y]) \\ &= [P,Q]. \end{aligned}$$

Hence $[P, Q] \in \chi(D_l)$. Thus, $F_\lambda(2\nu+3,4)-$Hsu structure satisfying equation (1.1) on M defines a $CR-$Structure.

Definition 2.1. Let $\bar{k}$ be the complementary distribution of R_eH to (M). We define a morphism of vector bundles $F : T(M) \to T(M)$, given by $F(X) = 0$ for all $X \in \chi(\bar{k})$, such that

$$FX = \frac{1}{2}\sqrt{-1}(-1)(P - \tilde{P}), \tag{2.11}$$

where $P = X + \sqrt{-1}(-1)Y \in \chi(H_P)$ and $\tilde{P}$ is complex conjugate of P.

Corollary 2.1. If $P = X + iY$ and $\tilde{P} = X - iY$ belong to H_P and $F(X) = \frac{1}{2}\sqrt{-1}$, $F(Y) = \frac{1}{2}(P+\tilde{P})$ and $F(-Y) = -\frac{1}{2}(P+\tilde{P})$, then $F(X) = -Y$, $F^2(X) = -X$ and $F(-Y) = -X$.

Theorem 2.8. If M has a $CR-$structure H, then $F^{2\nu+3} + \lambda^r F^4 = 0$ and consequently $F_\lambda(2\nu+3,4)-$Hsu structure is defined on M such that the distribution D_l and D_m coincide with $R_e(H)$ and $\bar{k}$ respectively.

Proof. Let M has a $CR-$structure. Then operating equation $F(X) = -Y$ by $(F^{2\nu} + \lambda^r F)$, we get

$$(F^{2\nu} + \lambda^r F)F(X) = (F^{2\nu} + \lambda^r F)(-Y).$$

On making use of corollary (2.1), the above equation becomes

$$\begin{aligned} (F^{2\nu+3} + \lambda^r F^4)(X) &= (F^{2\nu+1} + \lambda^r F^2)F^2(X) \\ &= (F^{2\nu+1} + \lambda^r F^2)(-X) \\ &= -(F^{2\nu-1} + \lambda^r)F^2(X) \\ &= -(F^{2\nu-1} + \lambda^r)(-X) \\ &= (F^{2\nu-1} + \lambda^r)(X). \end{aligned}$$

We continue simplifying in same manner and obtain

$$\left(F^{2\nu+3} + \lambda^r F^4\right) X = 0,$$

which is indeed $$F^{2\nu+3}(X) + \lambda^r F^4(X) = 0.$$

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