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On Finsler Spaces Satisfying the Condition $L^{m+1}C = \gamma^m$

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Abstract

In the year 1979, M. Matsumoto has discussed non-Riemannian Finsler spaces with vanishing T-Tensor. In the paper, M. Matsumoto has shown that if a Finsler space M^n satisfy $T-$condition i.e. $T_{hijk} = 0$, Then for such a Finsler space the function L^2C^2 of M^n is a function of position only (i.e. $L^2C^2 = f(x)$), where L is fundamental function and C^2 is the square of length of torsion tensor C_i. In continuity of the above paper F. Ikeda in the year 1984, studied Finsler spaces L^2C^2 as a function of x in detail. In the year 1991, Ikeda considered Finsler spaces satisfying the condition L^2C^2 as to non-zero constant, which is a stronger condition. One of the author T. N. Pandey in the year 2012 studied Finsler spaces taken L^2C^2 equal to some known function of x and y i.e. $L^2C^2 = f(x) + f(y)$.

In the present paper we shall consider the combination of L and C differently and taking $L^{m+1}C = \lambda^m$, where γ is m^{th} root metric.

1. Introduction

M. Matsumoto in the paper, [9] studied non-Riemannian Finsler spaces with the vanishing $T-$tensor, which are said to satisfy the $T-$condition (by $T-$condition we mean a Finsler space whose $T-$tensor vanishes), then the function L^2C^2 over M^n is reduced to a function of the position only (i.e., $L^2C^2 = f(x)$) where L is the fundamental function and C is the length of torsion vector C_i.

He has also quoted that if the metric tensor g_{ij} has a special form as $g_{ij} = q^{ls}_{ij}$ then the function L^2C^2 becomes zero (i.e. $L^2C^2 = 0$). Because in this case the $T-$condition satisfies automatically and $C_i = 0$.

In the continuity of the above paper F.Ikeda in the year 1984, studies Finsler spaces, L^2C^2 is a function of x in detail and come out with some interesting result specially for two and three dimensional Finsler spaces. Actually Ikeda was examine the equivalence of $T-$condition with $L^2C^2 = f(x)$. In the year 1991, F. Ikeda consider Finsler spaces satisfying the condition L^2C^2 equals to non-zero constant, which is stronger condition, then the condition imposed by him in the paper 1984.

An example of such a Finsler spaces with a constant function L^2C^2 is a two dimensional Berwald space. One of the author (T. N. Pandey), in the year 2012 [2] studied Finsler spaces taken LC equal to some known function of x and y, i.e. $LC = f(x) + g(x)$.

In the present paper, we shall consider combination of L and C differently and taking $L^{m+1}C = \gamma^m$, where γ is well known M^{th} root metric. For such a Finsler space if it is $C-$reducible it has been worked out under what condition $T-$Tensor vanishes. In the last section it has been worked out under what condition such a Finsler space ($L^{m+1}C = \gamma^m$) is a Landsberg space or Berwald space.

2. T-Tensor of a Finsler space with $L^{m+1}C = \gamma^m$

Let l_i, h_{ij} and C_{ijk} denote the unit vector, angular metric tensor and the (h) $hv-$torsion tensor respectively.

The well-known $T-$Tensor T_{ijkh} ([7], # equation (28.20)) has been defined by

$$T_{ijkh} = LC_{ijk|h} + C_{ijk}l_h + C_{jkh}l_i + C_{khi}l_j + C_{hij}l_k. \tag{2.1}$$

and the torsion Tensor C_i is given by $C_i = g^{jk}C_{ijk}$, where the symbol $|_h$ denote $v-$covariant differentiation and g^{jk} denote reciprocal of g_{jk}. We are considering a F^n whose torsion tensor is such that

$$L^{m+1}C = \gamma^m, \tag{2.2}$$

where $\gamma^m = a_{i_1 i_2 \ldots i_m} y^{i_1} y^{i_2} \ldots y^{i_m}$ is the m^{th} root metric.

Differentiating equation (2.2) with respect to y^h, we get

$$(m+1)L^m l_h C + L^{m+1}C_{;h} = m a_{h i_2 i_3 \ldots i_m} y^{i_2} y^{i_3} \ldots y^{i_m},$$

$$(m+1)Ll_h C + L^2 C_{;h} = \frac{m}{L^{m-1}} a_{h i_2 i_3 \ldots i_m} L^{m-1} l^{i_2} \ldots l^{i_m}. \qquad \left(\because \frac{y^i}{L} = l^i\right).$$

$$(m+1)LCl_h + L^2 C_{;h} = m a_h \tag{2.3}$$

where $C_{;h} = \frac{\partial C}{\partial y^h}$ and $a_h = a_{hi_2 i_3 \dots i_m} l^{i_2} \dots l^{i_m}$.

Now,

$$T_{ijkh} = LC_{ijk|h} + C_{ijk} l_h + C_{jkh} l_i + C_{khi} l_j + C_{hij} l_k.$$

Contract with g^{jk} and summing with respect to j and k

$$T_{ih} = LC_i|_h + C_i l_h + C_h l_i + C^j_{hi} l_j + C^k_{ih} l_k,$$

$$C^i T_{ih} = LC_i|_h . C^i + C^2 l_h. \tag{2.4}$$

Now, differentiating the equation $C^2 = g^{ij} C_i C_j$ with respect to y^h

$$2CC_{;h} = g^{ij} \frac{\partial C_i}{\partial y^h} C_j + g^{ij} \frac{\partial C_j}{\partial y^h} C_i,$$

$$C_{;h} = \frac{C^i C_{i|h}}{C}.$$

From (2.3),

$$(m+1) L l_h C + L^2 \frac{C^i C_i|_h}{C} = m a_h,$$

$$LC^i C_i|_h = C \left[\frac{a_h}{L} m - (m+1) l_h C \right]. \tag{2.5}$$

In the virtue of equation (2.4) and equation (2.5), we obtain

$$C^i T_{ih} = C \left[m \frac{a_h}{L} - (m+1) l_h C \right] + C^2 l_h,$$

$$C^i T_{ih} = mC \left(\frac{a_h}{L} - C l_h \right) \tag{2.6}$$

Conversely, let

$$C^i T_{ih} = mC \left(\frac{a_h}{L} - C l_h \right),$$

$$LC^i C_i|_h + C^2 l_h = mC \left(\frac{a_h}{L} - C l_h \right),$$

$$LC_{;h} + (m+1) C l_h = m \left(\frac{a_h}{L} \right), \qquad \text{since} \qquad C^i C_i|_h = CC_{;h}$$

$$L^2 C_{;h} + (m+1) C L l_h = m a_h.$$

Multiplying both side by L^{m-1}, we get

$$(L^{m+1} C)_{;h} = m a_h L^{m-1}.$$

Multiplying both side by y^h and using Euler's theorem, we get

$$L^{m+1} C = a_h L^m l^h = a_{hi_2 i_3 \dots i_m} y^{i_2} y^{i_3} \dots y^{i_m} y^h,$$

$$L^{m+1} C = \gamma^m.$$

Thus, we have

Theorem (2.1). For a Finsler space (M^n, L) of dimension n, if torsion Tensor C is such as $L^{m+1}C = \gamma^m$. Then following relation $C^iT_{ih} = mC\big(\frac{a_h}{L} - Cl_h\big)$ holds good.

Next, for a two dimensional Finsler space the $T-$Tensor ([7], # equation (28.3))

$$T_{hijk} = I_{;2}m_hm_im_jm_k, \tag{2.7}$$

where $I_{;2} = L\frac{\partial I}{\partial y^i}m^i$, also $LC_{ijk} = Im_im_jm_k$ and $LC = I$ ([7],# equation (28.3)).

Writing $LC = I$ in equation (2.2) and differentiating with respect to y^i, we have

$$(L^mI)_{;i} = \gamma^m_{;i}$$

$$mIl_i + I_{;2}m_i = ma_i \qquad \text{since} \qquad \big(L\frac{\partial I}{\partial y^i} = I_{;2}m_i\big).$$

Contracting both sides by m^i and we have,

$$I_{;2} = mm^i\,a_i.$$

From (2.5), we have

$$T_{hijk} = ma_rm^rm_hm_im_jm_k. \tag{2.8}$$

Corollary (2.1). For a two dimensional Finsler space if $L^{m+1}C = \gamma^m$ and a_i is parallel to l_i, then $T_{hijk} = 0$.

Next, for a $C-$Reducible Finsler space the $T-$Tensor [7] can be written as,

$$T_{hijk} = \frac{LC^*}{(n-1)^2}\,\pi_{hijk}h_{hi}h_{jk}, \tag{2.9}$$

where $C^* = g^{ij}C_i|_j$ and π_{ijk} represent sum of cyclic permutation in the indices h, i, j, k.

Contracting equation (2.9) by g^{jk}, we get

$$T_{hi} = \frac{LC^*}{(n-1)}h_{hi}.$$

Using equation (2.4)

$$C^iT_{ih} = C^i\frac{L}{n-1}\,C^*h_{hi} = LC^*\frac{C_h}{n-1} = mC\big(\frac{a_h}{L} - Cl_h\big), \tag{2.10}$$

$$a_h = \frac{L}{m}\,\big[\frac{LC^*C_h}{C(n-1)} + mCl_h\big],$$

$$C^h a_h = \frac{L}{m}\left[\frac{LCC^*}{n-1}\right].$$

Corollary (2.2). For a $C-$Reducible Finsler space (M^n, L) with $L^{m+1}C = \gamma^m$ and a_i parallel to l_i, then

$$T_{hijk} = 0.$$

3. Landsberg space and Berwald spaces satisfying the condition $L^{m+1} C = \gamma^m$

Let us consider a Finsler space M^n, where C is such that

$$L^{m+1}C = \gamma^m.$$

Differentiating above equation with respect to y^i, we get

$$L^{m+1}C_{;i} + (m+1)L^m C l_i = m\gamma^{m-1}\gamma_i, \tag{3.1}$$

where $C_{;i} = \frac{\partial C}{\partial y^i}$ and $\gamma_i = \frac{\partial \gamma}{\partial y^i}$. Differentiating equation (3.1) with respect to y^j, we get

$$\begin{aligned}(m+1)L^m l_j C_{;i} + L^{m+1}C_{;i;j} + m(m+1)CL^{m-1}l_i l_j + (m+1)L^m C_{;j} l_i \\ + (m+1)CL^{m-1}h_{ij} = m(m-1)\gamma^{m-1}\gamma_i\gamma_j + m\gamma^{m-1}\gamma_{ij}.\end{aligned} \tag{3.2}$$

Using $h_{ij} = g_{ij} - l_i l_j$, we have

$$\begin{aligned}g_{ij} =& \frac{1}{(m+1)C}\left[\frac{m\gamma^{m-2}}{L^{m-1}}\{\gamma\gamma_{ij} + (m-1)\gamma_i\gamma_j\} - \{(m+1)L(C_{;i}l_j + C_{;j}l_i) \right. \\ & \left. + (m^2-1)Cl_i l_j + L^2\frac{\partial C_{;i}}{y^j}\}\right].\end{aligned} \tag{3.3}$$

Again differentiating equation (3.2) with respect to y^k, we get

$$\begin{aligned}C_{ijk} =& \frac{1}{2(m+1)CL^{m-1}}\Big[m\gamma^m\gamma_{ijk} + m(m-1)\gamma^{m-2}(\pi_{ijk}\gamma_i\gamma_{jk}) + m(m \\ & - 1)(m-2)\gamma^{m-3}\gamma_i\gamma_j\gamma_k - L^{m+1}C_{;i;j;k} - (m+1)L^m(\pi_{ijk}l_i C_{;j;k}) \\ & - (m+1)mL^{m-1}(\pi_{ijk}C_{;i}l_j l_k) - (m+1)L^{m-1}(\pi_{ijk}C_{;i}h_{jk}) \\ & - (m-1)(m+1)L^{m-2}C(\pi_{ijk}l_i h_{jk}) \\ & - m(m+1)(m-1)CL^{m-2}l_i l_j l_k\Big].\end{aligned} \tag{3.4}$$

Since

$$\gamma^m = a_{i_1 i_2 i_3 \ldots i_m}(x)\, y^{i_1} y^{i_2} \ldots y^{i_m}.$$

Differentiating above with respect to y^i and y^j and using obtained value in (3.4), then (3.4) becomes

$$\begin{aligned} C_{ijk} =& \frac{1}{2(m+1)CL^{m-1}}\Big[m(m-1)(m-2)L^{m-3}a_{ijk} - L^{m+1}C_{;i;j;k} \\ & - (m+1)L^{m}(\pi_{ijk}l_iC_{;j;k}) - m(m+1)L^{m-1}(\pi_{ijk}C_{;i}l_jl_k) \\ & - (m+1)L^{m-1}(\pi_{ijk}C_{;i}h_{jk}) - (m-1)(m+1)L^{m-2} \\ & C(\pi_{ijk}l_ih_jk) - m(m+1)(m-1)L^{m-2}Cl_il_jl_k\Big], \end{aligned} \tag{3.4$'$}$$

$$\begin{aligned} C_{ijk|h} =& \frac{1}{2(m+1)CL^{m-1}}\Big[m(m-1)(m-2)L^{m-3}a_{ijk|h} - L^{m+1}C_{;i;j;k|h} \\ & - (m+1)L^{m}(\pi_{ijk}l_iC_{;j;k|h}) - m(m+1)L^{m-1}(\pi_{ijk}C_{;i|h}l_jl_k) - (m \\ & + 1)L^{m-1}(\pi_{ijk}C_{;i|h}h_{jk}) - (m-1)(m+1)L^{m-2}C_{|h})(\pi_{ijk}l_ih_{jk}) \\ & - m(m-1)(m+1)L^{m-2}C_{|h}l_il_jl_k\Big] + \frac{1}{2(m+1)}\left(\frac{1}{CL^{m-1}}\right)_{|h} \\ & \Big[m(m-1)(m-2)L^{m-3}a_{ijk} - L^{m+1}C_{;i;j;k} - (m+1)L^{m}(\pi_{ijk}l_iC_{;j;k}) \\ & - m(m+1)L^{m-1}(\pi_{ijk}C_{;i}l_jl_k) - (m+1)L^{m-1}(\pi_{ijk}C_{;i}h_{jk}) - (m \\ & - 1)(m+1)L^{m-2}C(\pi_{ijk}l_ih_{jk}) - m(m-1)(m+1)CL^{m-2}l_il_jl_k\Big]. \end{aligned} \tag{3.5}$$

Contracting equation (3.5) by y^h, we get

$$P_{ijk} = C_{ijk|h}\, y^h$$

$$\begin{aligned} y^hC_{ijk|h} =& \frac{1}{2(m+1)CL^{m-1}}\Big[m(m-1)(m-2)L^{m-3}a_{ijk|0} - L^{m+1}C_{;i;j;k|0} \\ & - (m+1)L^{m}(\pi_{ijk}l_iC_{;j;k|0}) - m(m+1)L^{m-1}(\pi_{ijk}C_{;i|0}l_jl_k) \\ & - (m+1)L^{m-1}(\pi_{ijk}C_{;i|0}h_{jk}) - (m-1)(m+1)L^{m-2}C_{|0}(\pi_{ijk}l_ih_{jk}) \\ & - m(m-1)(m+1)L^{m-2}C_{|0}l_il_jl_k\Big] + \frac{1}{2(m+1)}\left(\frac{1}{CL^{m-1}}\right)_{|h} \\ & \Big[m(m-1)(m-2)L^{m-3}a_{ijk} - L^{m+1}C_{;i;j;k} - (m+1)L^{m}(\pi_{ijk}l_iC_{;j;k}) \\ & - m(m+1)L^{m-1}(\pi_{ijk}C_{;i}l_jl_k) - (m+1)L^{m-1}(\pi_{ijk}C_{;i}h_{jk}) \\ & - (m-1)(m+1)L^{m-2}C(\pi_{ijk}l_ih_{jk}) \end{aligned}$$

$$- m(m-1)(m+1)CL^{m-2}l_il_jl_k\Big] \tag{3.6}$$

where P_{ijk} is the (V) $hv-$torsion tensor. The symbol $|_h$ denotes the $h-$covarient differentiation and the index '0' means contraction by y^h.

If we put $C_{ijk|h} = 0$ and $P_{ijk} = 0$, respectively, then we obtain

$$\begin{aligned} &\frac{1}{2(m+1)CL^{m-1}}\Big[m(m-1)(m-2)L^{m-3}a_{ijk|h} - L^{m+1}C_{;i;j;k|h} \\ &- (m+1)L^m(\pi_{ijk}l_iC_{;j;k|h}) - m(m+1)L^{m-1}(\pi_{ijk}C_{;i|h}l_jl_k) \\ &- (m+1)L^{m-1}(\pi_{ijk}C_{;i|h}h_{jk}) - (m-1)(m+1)L^{m-2}C_{|h}(\pi_{ijk}l_ih_{jk}) \\ &- m(m-1)(m+1)L^{m-2}C_{|h}l_il_jl_k\Big] + \frac{1}{2(m+1)}\left(\frac{1}{CL^{m-1}}\right)_{|h} \\ &\Big[m(m-1)(m-2)L^{m-3}a_{ijk} - L^{m+1}C_{;i;j;k} - (m+1)L^m(\pi_{ijk}l_iC_{;j;k}) \\ &- m(m+1)L^{m-1}(\pi_{ijk}C_{;i}l_jl_k) - (m+1)L^{m-1}(\pi_{ijk}C_{;i}h_{jk}) \\ &- (m-1)(m+1)L^{m-2}C(\pi_{ijk}l_ih_{jk}) \\ &- m(m-1)(m+1)CL^{m-2}l_il_jl_k\Big] = 0. \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &\frac{1}{2(m+1)CL^{m-1}}\Big[m(m-1)(m-2)L^{m-3}a_{ijk|0} - L^{m+1}C_{;i;j;k|0} \\ &- (m+1)L^m(\pi_{ijk}l_iC_{;j;k|0}) - m(m+1)L^{m-1}(\pi_{ijk}C_{;i|0}l_jl_k) \\ &- (m+1)L^{m-1}(\pi_{ijk}C_{;i|0}h_{jk}) - (m-1)(m+1)L^{m-2}C_{|0}(\pi_{ijk}l_ih_{jk}) \\ &- m(m-1)(m+1)L^{m-2}C_{|0}l_il_jl_k\Big] + \frac{1}{2(m+1)}\left(\frac{1}{CL^{m-1}}\right)_{|h} \\ &\Big[m(m-1)(m-2)L^{m-3}a_{ijk} - L^{m+1}C_{;i;j;k} - (m+1)L^m(\pi_{ijk}l_iC_{;j;k}) \\ &- m(m+1)L^{m-1}(\pi_{ijk}C_{;i}l_jl_k) - (m+1)L^{m-1}(\pi_{ijk}C_{;i}h_{jk}) \\ &- (m-1)(m+1)L^{m-2}C(\pi_{ijk}l_ih_{jk}) \\ &- m(m-1)(m+1)CL^{m-2}l_il_jl_k\Big] = 0. \end{aligned} \tag{3.8}$$

Therefore, we have

Theorem (3.1). If torsion scalar C of $F^n = (M^n, L)$ satisfies the condition $L^{m+1}C = \gamma^m$, then necessary and sufficient condition for M^n to be a Berwald space is that the equation (3.7) holds good.

Theorem (3.2). If torsion scalar C of $F^n = (M^n, L)$ satisfies the condition $L^{m+1}C = \gamma^m$, then necessary and sufficient condition for M^n to be a Landsberg space is that the equation (3.8) holds good.

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