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Geometric Properties of Weakly Berwald Space with Some (α, β)-metric

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Abstract

The (α, β)-metric is a Finsler metric which is contstructed from a Riemannian metric α and a differential 1-form β. In this paper Finsler space with some (α, β)-metrics like $L = (\alpha + \beta)^2/\alpha$ and $L^2 = 2\alpha\beta$ becomes weakly Berwald spaces under some geometric and algebraic conditions.

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1. Introduction

The concept of (α, β)-metric was proposed by M. Matsumoto (1972) and investigated in detail by S. Kikuchi [11], C. Shibata [1], M. Hashiguchi and others [5]. The study of some well known metrics like Randers metric $\alpha + \beta$, Kropina metric α^2/β have greatly contributed to the growth of Finsler geometry and its applications to the theory of relativity. S.Basco [10] gave the definition of a weakly-Berwald space as another generalization of Berwald space. The authors Il-Yong Lee and Myung-Han Lee [3] have studied on weakly Berwald spaces of special (α, β)-metrics. In this paper we extended the study to some other (α, β)-metrics.

2. Preliminaries

By a Finsler space, we mean a triple $F^n = (M, D, L)$, where M denotes n-dimensional differentiable manifold, D is an open subset of a tangent vector bundle TM endowed with the differentiable structure induced by the differentiable structure of the manifold TM and $L : D \longrightarrow R$ is a differentiable mapping having the following properties:

$i)\quad L(x,y) > 0,\ \text{ for } (x,y) \in D,$

$ii)\quad L(x,\lambda y) = |\lambda| L(x,y), \quad \text{for any } (x,y) \in D \text{ and } \lambda \in R, \text{ such that } (x,\lambda y) \in D,$

$iii)\quad \text{The } d-\text{tensor field } g_{ij}(x,y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2, \quad (x,y) \in D,$

where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$, is non-degenerate on D.

The metric tensor g_{ij} and Cartan's C-tensor C_{ijk} are given by [2]

$$\begin{aligned} g_{ij} &= \dot{\partial}_i\dot{\partial}_j L^2/2, \quad g^{ij} = (g_{ij})^{-1}, \\ C_{ijk} &= \frac{1}{2}\dot{\partial}_k g_{ij}, \quad C^k_{ij} = \frac{1}{2} g^{kh} C_{ihj}. \end{aligned}$$

We use the following [2]

$(a)\quad l_i = \dot{\partial}_i L, \quad l^i = y^i/L,$

$(b)\quad h_{ij} = g_{ij} - l_i l_j,$

$(c)\quad \gamma^i_{jk} = \frac{1}{2} g^{ih}\left\{\partial_k g_{jh} + \partial_j g_{kh} - \partial_h g_{jk}\right\},$

$(d)\quad G^i = \frac{1}{2}\gamma^i_{jk} y^j y^k, \quad G^i_j = \dot{\partial}_j G^i, \quad G^i_{jk} = \dot{\partial}_k G^i_j, \quad G^i_{jkl} = \dot{\partial}_l G^i_{jk}.$

A Finsler metric $L(x,y)$ is called an (α,β)-metric $L(\alpha,\beta)$ if L is a positively homogeneous function of α and β of degree one, where $\alpha^2 = a_{ij}(x) y^i y^j$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form on M^n.

We define G_i as [3]

$$G_i = \{y^h(\partial_h \dot{\partial}_i L^2) - \partial_i L^2\}/4,$$

and $G^i = g^{ij} G_j$, (g^{ij}) is the inverse matrix of (g_{ij}) and the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$. A Berwald space is a Finsler space which satisfies the condition $G^h_{ijk} = 0$, that is, whose coefficients G^h_{ij} of the Berwald connection are functions of the position (x^i) alone. Therefore it satisfies the condition $y_h G^h_{ijk} = 0$, so that $2G^i = G^i_{jk} y^j y^k$ are homogeneous polynomials in (y^i) of degree two. So $D^{ij} = G^i y^j - G^j y^i$ are homogeneous polynomials in (y^i) of degree three. Thus we consider the notion of Landsberg spaces, Douglas spaces, weakly-Berwald spaces as three generalizations of Berwald spaces. Thus we state that a Finsler space is said to be weakly Berwald space if it satisfies the condition $G_{ij} = 0$.

A Finsler space with an (α, β)-metric is a weakly Berwald space if and only if $B^m_m = \partial B^m/\partial y^m$ is a one-form [3] i.e, $B^m_m = \partial B^m/\partial y^m$ is a homogeneous polynomial in (y^i) of degree one. In other words, a Finsler space with an (α, β)-metric is a Berwald space, if and only if B^m are homogeneous polynomials in (y^i) of degree two. In this paper we find that the conditions for the Finsler spaces with the metric $L = (\alpha + \beta)^2/\alpha$ and $L^2 = 2\alpha\beta$ be weakly Berwald spaces.

3. Weakly-Berwald space with respect to (α, β)-metric

In this section, the symbol $(;)$ stands for h-covariant derivation with respect to the Riemannian connection in the space (M^n, α) and γ^i_{jk} stands for the Christoffel symbols in the space (M^n, α).

We use the following notations [3]

$$\begin{aligned}
&(a) \quad r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i}), \quad r^i_j = a^{ih} r_{hj}, \quad r_j = b_i r^i_j, \\
&(b) \quad s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i}), \quad s^i_j = a^{ih} s_{hj}, \quad s_j = b_i s^i_j, \\
&(c) \quad b^i = a^{ih} b_h, \quad b^2 = b^i b_i.
\end{aligned}$$

Define the functions G^m of F^n with an (α, β)-metric as [9][3];

$$2G^m = \gamma^m_{00} + 2B^m, \tag{3.1}$$

where

$$B^m = (E^*/\alpha)y^m + (\alpha L_\beta/L_\alpha)s^m_0 - (\alpha L_{\alpha\alpha}/L_\alpha)C^*\{(y^m/\alpha) - (\alpha/\beta)b^m\}, \tag{3.2}$$

we put

$$\begin{aligned}
E^* &= (\beta L_\beta/L)C^*, \\
C^* &= \{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)\}/\{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})\}, \\
\gamma^2 &= b^2\alpha^2 - \beta^2.
\end{aligned} \tag{3.3}$$

Differentiating (3.2) by y^n and contracting m and n in the obtained equation, we have

$$\begin{aligned}
B^m_m = &\left\{\dot\partial_m\left(\frac{\beta L_\beta}{\alpha L}\right)y^m + \frac{n\beta L_\beta}{\alpha L} - \dot\partial_m\left(\frac{\alpha L_{\alpha\alpha}}{L_\alpha}\right)\left(\frac{\beta y^m - \alpha^2 b^m}{\alpha\beta}\right)\right\}C^* \\
&- \frac{\alpha L_{\alpha\alpha}}{L_\alpha}\left\{\dot\partial_m\left(\frac{1}{\alpha}\right)y^m + \frac{1}{\alpha}\delta^m_m - \dot\partial_m\left(\frac{\alpha}{\beta}\right)b^m\right\}C^* \\
&+ \left(\frac{\beta L_\alpha L_\beta - \alpha L L_{\alpha\alpha}}{\alpha L L_\alpha}\right)(\dot\partial_m C^*)y^m + \left(\frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}\right)(\dot\partial_m C^*)b^m + \dot\partial_m\left(\frac{\alpha L_\beta}{L_\alpha}\right)s^m_0.
\end{aligned} \tag{3.4}$$

Since $L = L(\alpha, \beta)$ is a positively homogeneous function of α and β of degree one, we have

$$L_\alpha \alpha + L_\beta \beta = L, \quad L_{\alpha\alpha}\alpha + L_{\alpha\beta}\beta = 0,$$
$$L_{\beta\alpha}\alpha + L_{\beta\beta}\beta = 0, \quad L_{\alpha\alpha\alpha}\alpha + L_{\alpha\alpha\beta}\beta = -L_{\alpha\alpha}.$$

Using the above and the homogeneity of (y^i), we obtain the following equations

$$\dot{\partial}_m \left(\frac{\beta L_\beta}{\alpha L} \right) y^m = - \frac{\beta L_\beta}{\alpha L}, \tag{3.5}$$

$$\dot{\partial}_m \left(\frac{\alpha L_{\alpha\alpha}}{L_\alpha} \right) \left(\frac{\beta y^m - \alpha^2 b^m}{\alpha\beta} \right) = \frac{\gamma^2}{(\beta L_\alpha)^2} \{ L_\alpha L_{\alpha\alpha} + \alpha L_\alpha L_{\alpha\alpha\alpha} - \alpha (L_{\alpha\alpha})^2 \}, \tag{3.6}$$

$$\left\{ \dot{\partial}_m \left(\frac{1}{\alpha} \right) y^m + \frac{1}{\alpha} \delta^m_m - \dot{\partial}_m \left(\frac{\alpha}{\beta} \right) b^m \right\} = \frac{1}{\alpha\beta^2} \{ \gamma^2 + (n-1)\beta^2 \}, \tag{3.7}$$

$$(\dot{\partial}_m C^*) y^m = 2C^*, \tag{3.8}$$

$$\begin{aligned} (\dot{\partial}_m C^*) b^m = \frac{1}{2\alpha\beta\Omega^2} [\Omega \{ \beta(\gamma^2 + 2\beta^2) W + 2\alpha^2\beta^2 L_\alpha r_0 - \alpha\beta\gamma^2 L_{\alpha\alpha} r_{00} \\ - 2\alpha(\beta^3 L_\beta + \alpha^2\gamma^2 L_{\alpha\alpha}) s_0 \} - \alpha^2 \beta W \{ 2b^2\beta^2 L_\alpha - \gamma^4 L_{\alpha\alpha\alpha} \\ - b^2 \alpha\gamma^2 L_{\alpha\alpha} \}], \end{aligned} \tag{3.9}$$

$$\dot{\partial}_m \left(\frac{\alpha L_\beta}{L_\alpha} \right) s^m_0 = \frac{\alpha^2 L L_{\alpha\alpha} s_0}{(\beta L_\alpha)^2}, \tag{3.10}$$

where

$$W = (r_{00} L_\alpha - 2\alpha s_0 L_\beta),$$

$$\Omega = (\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}), \text{ provided } that \ \Omega \neq 0, \tag{3.11}$$

$$Y_i = a_{ir} y^r, \quad s_{00} = 0, \quad b^r s_r = 0, \quad a^{ij} s_{ij} = 0.$$

Using (3.5), (3.6), (3.7), (3.8), (3.9) and (3.10) in (3.4), we obtain

$$\begin{aligned} B^m_m = \frac{1}{2\alpha L (\beta L_\alpha)^2 \Omega^2} \{ 2\Omega^2 A C^* + 2\alpha L \Omega^2 B s_0 \\ + \alpha^2 L L_\alpha L_{\alpha\alpha} (C r_{00} + D s_0 + E r_0) \}, \end{aligned} \tag{3.12}$$

where

$$
\begin{aligned}
A &= (n+1)\beta^2 L_\alpha(\beta L_\alpha L_\beta - \alpha L L_{\alpha\alpha}) \\
&\quad + \alpha\gamma^2 L\{\alpha(L_{\alpha\alpha})^2 - 2L_\alpha L_{\alpha\alpha} - \alpha L_\alpha L_{\alpha\alpha\alpha}\}, \\
B &= \alpha^2 L L_{\alpha\alpha}, \\
C &= \beta\gamma^2\{-\beta^2(L_\alpha)^2 + 2b^2\alpha^3 L_\alpha L_{\alpha\alpha} - \alpha^2\gamma^2(L_{\alpha\alpha})^2 + \alpha^2\gamma^2 L_\alpha L_{\alpha\alpha\alpha}\}, \\
D &= 2\alpha\{\beta^3(\gamma^2 - \beta^2)L_\alpha L_\beta - \alpha^2\beta^2\gamma^2 L_\alpha L_{\alpha\alpha} \\
&\quad - 2\alpha\beta\gamma^2(\gamma^2 + 2\beta^2)L_\beta L_{\alpha\alpha} - \alpha^3\gamma^4(L_{\alpha\alpha})^2 - \alpha^2\beta\gamma^4 L_\beta L_{\alpha\alpha\alpha}\}, \\
E &= 2\alpha^2\beta^2 L_\alpha \Omega.
\end{aligned} \tag{3.13}
$$

Thus we conclude that

Theorem 3.1. [3] The necessary and sufficient condition for a Finsler space F^n with an (α, β)-metric to be a weakly-Berwald space is that $G^m_m = \gamma^m_{0m} + B^m_m$ and B^m_m is a homogeneous polynomial in (y^m) of degree one, where B^m_m is given by (3.12), provided that $\Omega \neq 0$.

Here we use the following;

Lemma 3.1. [6] If $\alpha^2 \equiv 0 \pmod{\beta}$, that is, $a_{ij}(x)y^i y^j$ contains $b_i(x)y^i$ as a factor, then the dimension is equal to two and b^2 vanishes. In this case we have $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.

4. Finsler space with the metric $L = (\alpha + \beta)^2/\alpha$

Consider the metric

$$
L(\alpha, \beta) = \frac{(\alpha + \beta)^2}{\alpha}. \tag{4.1}
$$

From (4.1), we have

$$
\begin{aligned}
L_\alpha &= (\alpha^2 - \beta^2)/\alpha^2, \\
L_\beta &= 2(\alpha + \beta)/\alpha, \\
L_{\alpha\alpha} &= 2\beta^2/\alpha^3, \\
L_{\alpha\alpha\alpha} &= -6\beta^2/\alpha^4.
\end{aligned} \tag{4.2}
$$

Substituting (4.2) in B^m, we have

$$B^m = \frac{(\alpha^2-\beta^2)r_{00} - 4\alpha^2(\alpha+\beta)s_0}{(1+2b^2)\alpha^2-3\beta^2}\left\{\left(\frac{1}{\alpha+\beta} - \frac{\beta}{\alpha^2-\beta^2}\right)y^m + \frac{\alpha^2}{\alpha^2-\beta^2}b^m\right\} + \frac{2\alpha^2(\alpha+\beta)}{\alpha^2-\beta^2}s_0^m, \tag{4.3}$$

and again substituting (4.2) into (3.13), (3.3) and (3.11) in respective quantities, we obtain

$$\begin{aligned}
A &= \frac{2(n+1)\beta^3(\alpha^2-\beta^2)(\alpha+\beta)[\alpha(\alpha-\beta)-2\beta^2]}{\alpha^5} + \frac{2\gamma^2\beta^2(\alpha+\beta)^2(\alpha^2+\beta^2)}{\alpha^5},\\
B &= \frac{2\beta^2(\alpha+\beta)^2}{\alpha^2},\\
C &= \frac{\beta^3\gamma^2}{\alpha^4}\{-(\alpha^2-\beta^2)^2 + 4b^2\alpha^2(\alpha^2-\beta^2) - 2\gamma^2(3\alpha^2-\beta^2)\},\\
D &= \frac{4\beta^3}{\alpha^2}\{(\gamma^2-\beta^2)(\alpha^2-\beta^2)(\alpha+\beta) - \beta\gamma^2(\alpha^2-\beta^2)\\
&\quad -4\gamma^2(\gamma^2+2\beta^2)(\alpha+\beta) + 2\gamma^4(3\alpha+2\beta)\},\\
E &= \frac{2\beta^4(\alpha^2-\beta^2)[(1+2b^2)\alpha^2-3\beta^2]}{\alpha^2},\\
\Omega &= \frac{\beta^2[(1+2b^2)\alpha^2-3\beta^2]}{\alpha^2},\\
W &= \frac{1}{\alpha^2}[(\alpha^2-\beta^2)r_{00} - 4\alpha^2(\alpha+\beta)s_0],\\
C^* &= \frac{\alpha[(\alpha^2-\beta^2)r_{00} - 4\alpha^2(\alpha+\beta)s_0]}{2\beta[(1+2b^2)\alpha^2-3\beta^2]}.
\end{aligned} \tag{4.4}$$

Substituting (4.4) into (3.12), we get

$$\begin{aligned}
&\{(1+2b^2)\alpha^{10}\beta + 2(7b^2-1)\alpha^4\beta^7 + 2(7+5b^2)\alpha^6\beta^5 - 7(1+2b^2)\alpha^8\beta^3 + 9\beta^{11}\\
&-3(5+4b^2)\alpha^2\beta^9 + 2(1+2b^2)\alpha^9\beta^2 + 4(11+13b^2)\alpha^5\beta^6 - 16(1+2b^2)\alpha^7\beta^4 + 18\alpha\beta^{10}\\
&-24(2+b^2)\alpha^3\beta^8\}B^m_m - \{(a_4 - 214 - b^4)\alpha^6\beta^4 + [a_5 + 2(b^4-5b^2+1)]\alpha^4\beta^6\\
&+[a_7 + (10b^2-2b^4+1)]\alpha^8\beta^2 + [(8+b^2)-a_9]\alpha^2\beta^8 + 6n\beta^{10} + [a_1 - 2b^2(1+2b^2)]\alpha^9\beta\\
&+[2(11b^2+1)-a_2]\alpha^7\beta^3 + [a_3 + 2(2b^4-11b^2-9)]\alpha^5\beta^5 + 3(3n-1)\alpha\beta^9\\
&+[2(11+b^2)-a_8]\alpha^3\beta^7\}r_{00} - \{[2(1+2b^2)(1+6b^2) - b_1]\alpha^{10}\beta + [24b^2(1+2b^2)\\
&-4(4b^4+36b^2+5) - b_2]\alpha^8\beta^3 + [2(1+2b^2)(2b^2-35) + 18 + 4(26b^2+37) - b_3]\alpha^6\beta^5\\
&-[18+12(2n+1)]\alpha^2\beta^9 + [b_8 + 24(4-b^2) + 4(8b^2-23)]\alpha^4\beta^7
\end{aligned}$$

$$+[8(1+2b^2)(2b^2-5)-b_4-2(9b^2-4b^4+1)]\alpha^7\beta^4+[4(22+29b^2)-b_5$$
$$+24(1-4b^2)]\alpha^5\beta^6+[8(1+2b^2)-b_7-4(13b^2+2)]\alpha^9\beta^2+[b_9+24]\alpha^3\beta^8\}s_0$$
$$-\{2(1+2b^2)\alpha^{10}\beta-4(2+b^2)\alpha^8\beta^3+4(1-b^2)\alpha^6\beta^5-6\alpha^2\beta^9+2(4+2b^2)\alpha^4\beta^7$$
$$+4(1+2b^2)\alpha^9\beta^2+4(7+2b^2)\alpha^5\beta^6-12\alpha^3\beta^8-4(5+4b^2)\alpha^7\beta^4\}r_0=0, \qquad (4.5)$$

where

$$a_1=(1+2b^2)(1+2b^2+n),$$
$$a_2=(1+2b^2)(5n+7)+3(1+2b^2+n),$$
$$a_3=(1+2b^2)(3n+1)-3(2b^2-4n-6)-(1+2b^2)(2b^2-4n-6)+3(1+2b^2+n),$$
$$a_4=(1+2b^2)(b^2-2n-4)-3(2b^2-1)-(4b^4-1)+3b^2,$$
$$a_5=(1+2b^2)(2n+1)-3(b^2-2n-4)-(1+2b^2)(b^2-2n-4)+3(2b^2-1),$$
$$a_6=(1+2b^2)b^2,$$
$$a_7=4b^4-1+3b^2-(1+2b^2)b^2,$$
$$a_8=3(3n+1)+(1+2b^2)(3n+1)-3(2b^2-2(2n+3)),$$
$$a_9=3(2n+1)+(1+2b^2)(2n+1)-3(b^2-2(n+2)).$$

$$b_1=4(1+2b^2)(1+2b^2+n)+4b^2(1+2b^2),$$
$$b_2=4(1+2b^2)(2b^2-2(2n+3))-12(1+2b^2+n)+4(4b^4-1-3b^2),$$
$$b_3=4(1+2b^2)(3n+1)-12(2b^2-2(2n+3))+4(1+2b^2)(b^2-2n-4)-12(2b^2-1),$$
$$b_4=4(1+2b^2)(b^2-2n-4)-12(2b^2-1)+4(1+2b^2)(2b^2-2(2n+3))-3(2b^2+n+1),$$
$$b_5=4(1+2b^2)(2n+1)-12(b^2-2n-4)+4(1+2b^2)(3n+1)-3(2b^2-4n-6),$$
$$b_6=4(1+2b^2)b^2,$$
$$b_7=4(4b^4-1-3b^2)+4(1+2b^2)(1+2b^2+n),$$
$$b_8=12(3n+1)+4(1+2b^2)(2n+1)-12(b^2-2n-4)),$$
$$b_9=12(2n+1)+12(3n+1).$$

Now we assume that F^n is a weakly Berwald space, then B^m_m is $hp(1)$. Since α is irrational in (y^i), the equation (4.5) is divided into two equations as follows,

$$F_1B^m_m + \beta G_1r_{00} + \alpha^2H_1s_0 + \alpha^2I_1r_0 = 0, \tag{4.6}$$

$$\beta F_2B^m_m + G_2r_{00} + \alpha^2\beta H_2s_0 + \alpha^2\beta I_2r_0 = 0, \tag{4.7}$$

where

$$\begin{aligned}F_1 &= (1+2b^2)\alpha^{10} + 2(7b^2-1)\alpha^4\beta^6 + 2(7+5b^2)\alpha^6\beta^4 - 7(1+2b^2)\alpha^8\beta^2 + 9\beta^{10}\\ &\quad - 3(5+4b^2)\alpha^2\beta^8,\\ F_2 &= +2(1+2b^2)\alpha^8 + 4(11+13b^2)\alpha^4\beta^4 - 16(1+2b^2)\alpha^6\beta^2 + 18\beta^8\\ &\quad - 24(2+b^2)\alpha^2\beta^6,\\ G_1 &= -\{[a_4 - 2(4-b^4)]\alpha^6\beta^2 + [a_5 + 2(b^4 - 5b^2 + 1)]\alpha^4\beta^4\\ &\quad + [a_7 + (10b^2 - 2b^4 + 1)]\alpha^8 + [(8+b^2) - a_9]\alpha^2\beta^6 + 6n\beta^8\},\\ G_2 &= -\{[a_1 - 2b^2(1+2b^2)]\alpha^8 + [2(11b^2+1) - a_2]\alpha^6\beta^2\\ &\quad + [a_3 + 2(2b^4 - 11b^2 - 9)]\alpha^4\beta^4 + 3(3n-1)\beta^8 + [2(11+b^2) - a_8]\alpha^2\beta^6\},\\ H_1 &= -\{[2(1+2b^2)(1+6b^2) - b_1]\alpha^8 + [24b^2(1+2b^2) - 4(4b^4 + 36b^2 + 5)\\ &\quad - b_2]\alpha^6\beta^2 + [2(1+2b^2)(2b^2-35) + 18 + 4(26b^2+37) - b_3]\alpha^4\beta^4\\ &\quad - [18 + 12(2n+1)]\beta^8 + [b_8 + 24(4-b^2) + 4(8b^2-23)]\alpha^2\beta^6\},\\ H_2 &= -\{[8(1+2b^2)(2b^2-5) - b_4 - 2(9b^2 - 4b^4 + 1)]\alpha^4\beta^2 + [4(22+29b^2) - b_5\\ &\quad + 24(1-4b^2)]\alpha^2\beta^4 + [8(1+2b^2) - b_7 - 4(13b^2+2)]\alpha^6 + [b_9 + 24]\beta^6\},\\ I_1 &= -\{2(1+2b^2)\alpha^8 - 4(2+b^2)\alpha^6\beta^2 + 4(1-b^2)\alpha^4\beta^4 - 6\beta^8 + 2(4+2b^2)\alpha^2\beta^6\},\\ I_2 &= -\{4(1+2b^2)\alpha^6 + 4(7+2b^2)\alpha^2\beta^4 - 12\beta^6 - 4(5+4b^2)\alpha^4\beta^2\}.\end{aligned}$$

Eliminating B^m_m from these equations, we obtain

$$Rr_{00} + \alpha^2\beta Ss_0 + \alpha^2\beta Tr_0 = 0, \tag{4.8}$$

where

$$R = \beta^2F_2G_1 - F_1G_2, \quad S = F_2H_1 - F_1H_2, \quad T = F_2I_1 - F_1I_2.$$

And (4.8) implies

$$(R/\alpha^2\beta)r_{00} + Ss_0 + Tr_0 = 0. \tag{4.9}$$

Since only the term $\epsilon_1\alpha^{16}$ of Ss_0 in (4.9) does not contain β, we must have $hp(16)V_{16}$, such that

$$\alpha^{16}s_0 = \beta V_{16}. \tag{4.10}$$

Here

$\epsilon_1 = (1+2b^2)\{4[(1+2b^2)(1+6b^2) - b_1] - [8(1+2b^2) - b_7 - 4(13b^2+2)]\}.$

First consider that $\alpha^2 \not\equiv 0 \pmod{\beta}$ and $b^2 \neq 0$. (4.10) shows the existence of a function $k(x)$ satisfying $V_{16} = k\alpha^{16}$, and hence $s_0 = k\beta$, (4.9) reduces to

$$(R/\alpha^2\beta)r_{00} + Sk\beta + Tr_0 = 0,$$

implies

$$Rr_{00} + Sk\alpha^2\beta^2 + \alpha^2\beta Tr_0 = 0.$$

Only the term $-(1+2b^2)[a_1 - 2b^2(1+2b^2)]\alpha^{18}r_{00}$ of the above does not contain β. Thus there exist $hp(19)U_{19}$ satisfying $-(1+2b^2)(a_1 - 2b^2(1+2b^2))\alpha^{18}r_{00} = \beta U_{19}$. It is a contradiction, which implies $k = 0$. Hence we obtain $s_0 = 0; s_j = 0$. (4.8) becomes

$$Rr_{00} + \alpha^2\beta Tr_0 = 0. \tag{4.11}$$

Only the term $27(n+1)\beta^{18}r_{00}$ of (4.11) seemingly does not contain α^2, and hence we must have $hp(18)V_{18}$ such that $\beta^{18}r_{00} = \alpha^2 V_{18}$. From $\alpha^2 \not\equiv 0 \pmod{\beta}$ there exist a function $f(x)$ such that

$$r_{00} = \alpha^2 f(x); \quad r_{ij} = a_{ij}f(x). \tag{4.12}$$

Transvecting above by $b^i y^j$, we have

$$r_0 = \beta f(x); \quad r_j = b_j f(x). \tag{4.13}$$

Substituting (4.12) and (4.13) into (4.11), we have

$$f(x)(R + \beta^2 T) = 0. \tag{4.14}$$

Assume that $f(x) \neq 0$, from (4.14) we get

$$R + \beta^2 T = 0,$$

the term $-(1+2b^2)(a_1 - 2b^2(1+2b^2))\alpha^{18}$ of the above does not contain β. Thus there exist $hp(17)V_{17}$ satisfying $-(1+2b^2)(a_1 - 2b^2(1+2b^2))\alpha^{18} = \beta V_{17}$, where V_{17} is $hp(17)$ this implies $V_{17} = 0$, provided that $b^2 \neq 0$. Hence $f(x) = 0$ must hold and we obtain

$$r_{00} = 0; \quad r_{ij} = 0 \textit{ and } r_0 = 0; \quad r_j = 0.$$

Conversely, substituting $r_{00} = 0$, $s_0 = 0$ and $r_0 = 0$ into (4.5), we have $B^m_m = 0$. That is, the Finsler space with (4.1) is a weakly-Berwald space.

On the other hand, we suppose that the Finsler space with (4.1) be a Berwald space. Then we have $r_{00} = 0$, $s_0 = 0$ and $r_0 = 0$, because the space is weakly Berwald space from the above discussion. Substituting the above into (4.3), we have $B^m = 0$, that is, the Finsler space with (4.1) is a Berwald space. Hence $s_{ij} = 0$ hold good.

Now consider $\alpha^2 \equiv 0 \pmod{\beta}$, Lemma 3.1 shows that $n = 2$, $b^2 = 0$ and $\alpha^2 = \beta\delta$, $\delta = d_i(x)y^i$. From these conditions (4.8) is rewritten in the form

$$R' r_{00} + \beta\delta S' s_0 = 0, \tag{4.15}$$

where

$$\begin{aligned}
R' &= 216\beta^9 - (432 + a_9)\beta^8\delta + 2(9a_5 + 24a_9 + 128)\beta^7\delta^2 + 2(9a_4 - 22a_9 - a_8 \\
&\quad - 12)\beta^6\delta^3[2(22a_5 + 9a_7 - 24a_4 + 8a_9 + 184) - (365 - 14a_8 - 2a_3)]\beta^5\delta^4 \\
&\quad + [2(22a_4 - 8a_5 - 24a_7 - a_9 - 208) - (7a_8 + 14a_3 + 2a_2 - 413)]\beta^4\delta^5 \\
&\quad [2(a_5 + 14a_7 - 8a_4 + 80) + (7a_3 + 14a_2 + 2a_1 + a_8 - 176)]\beta^3\delta^6 - a_1\delta^9 + [2 \\
&\quad (a_7 + 7) - (2 - a_2 - 7a_1)]\beta\delta^8 + [2(a_4 - 8) - (14a_1 + a_3 + 7a_2 - 32)]\beta^2\delta^7, \\
S' &= [2(2 - b_1) - b_7]\delta^8 - [2(20 + b_2) + 16(2 - b_1) - (42 + b_4) + 7b_7]\beta\delta^7 \\
&\quad + [44(96 - b_3) - 3588 - 16(4 + b_8) + 18(2 - b_1) + 48(20 + b_2) - 2(42 + b_4) \\
&\quad - 14(112 - b_5) + 7(b_9 + 24) - 15b_7]\beta^4\delta^4 + [2(b_8 + 4) - 44(20 + b_2) \\
&\quad + 2(96 - b_3) - 48(2 - b_1) + 14(42 + b_4) - (b_9 + 24) + 7(112 - b_5) \\
&\quad - 2b_7]\beta^3\delta^5 + [2(96 - b_3) + 44(2 - b_1) + 44(b_8 + 4) + 16(20 + b_2) \\
&\quad - 7(42 + b_4) - (112 - b_5) + 14b_7]\beta^2\delta^6 + [2(112 - b_5) - 14(b_9 + 24) \\
&\quad - 15(42 + b_4) - 48(96 - b_3) - 18(20 + b_2) + 9b_7]\beta^5\delta^3 + [15(112 - b_5) \\
&\quad + 2(b_9 + 24) + 9(42 + b_4) - 48(b_8 + 4)]\beta^6\delta^2 + [15(b_9 + 24) - 9(112 - b_5) \\
&\quad + 18(b_8 + 4) + 3644]\beta^7\delta - [1404 + 9(b_9 + 24)]\beta^8.
\end{aligned}$$

Since only the term $216\beta^9 r_{00}$ of $R' r_{00} + \beta\delta S' s_0$ in (4.15) seemingly does not contain δ, we must have $hp(1)V_1$ such that $r_{00} = \delta V_1$. We have $s_0 = 0$; $s_j = 0$, now (4.15) becomes

$$R' r_{00} = 0, \tag{4.16}$$

which implies

$$r_{00} = 0; \quad r_{ij} = 0 \quad \textit{and} \quad r_0 = 0; \quad r_j = 0.$$

Conversely from $r_{00} = 0$, $r_0 = 0$ and $s_0 = 0$ we have $B^m_m = 0$. Thus the space with (4.1) is weakly-Berwald space. Thus we state that

Theorem 4.2. A Finsler space with the metric (4.1) is weakly Berwald space if and only if the following conditions holds;

(1) $\alpha^2 \not\equiv 0(\mod \beta)$ implies $r_{ij} = 0$ and $s_j = 0$.
(2) $\alpha^2 \equiv 0(\mod \beta)$ implies $n = 2$, $b^2 = 0$ and $r_{ij} = 0$, $s_j = 0$ are satisfied, where $\alpha^2 = \beta\delta$, $\delta = d_i y^i$.

5. Finsler space with the metric $L^2 = 2\alpha\beta$

In this section we consider the Finsler space with the metric

$$L^2 = 2\alpha\beta. \tag{5.1}$$

From (5.1), we have

$$L_\alpha = \beta/L, \quad L_\beta = \alpha/L, \quad L_{\alpha\alpha} = -\beta^2/L^3, \quad L_{\alpha\alpha\alpha} = 3\beta^3/L^5. \tag{5.2}$$

Using (5.1), (3.13), (3.3), (3.11) and (3.12), we obtain

$$FB^m_m - \beta G r_{00} + \alpha^2 H s_0 + \alpha^2 I r_0 = 0, \tag{5.3}$$

where

$$\begin{aligned} F &= 9\beta^4 - 6b^2\alpha^2\beta^2 + b^4\alpha^4, \\ G &= 3n\beta^2 - (n-2)b^2\alpha^2, \\ H &= 3(2n+1)\beta^2 - (2n-3)b^2\alpha^2, \\ I &= 3\beta^2 - b^2\alpha^2. \end{aligned}$$

Now suppose that F^n be a weakly Berwald space, that is, B^m_m is $hp(1)$. Only the term $9\beta^4 B^m_m - 3n\beta^3 r_{00}$ of (5.3) does not contain α^2, and hence we must have $hp(3) V_3$ satisfying $\beta^3(9\beta B^m_m - 3n r_{00}) = \alpha^2 V_3$. Now we assume that $\alpha^2 \not\equiv 0(\mod \beta)$, above reduces to

$$9\beta B^m_m - 3n r_{00} = k\alpha^2, \tag{5.4}$$

with $V_3 = k\beta^3$. Further (5.4) implies that

$$B^m_m = \frac{k\alpha^2 + 3n r_{00}}{9\beta}. \tag{5.5}$$

Then (5.3) reduces to

$$F\left(\frac{k\alpha^2 + 3n r_{00}}{9\beta}\right) - \beta G r_{00} + \alpha^2 H s_0 + \alpha^2 I r_0 = 0,$$

implies

$$kF + 3nF' r_{00} - 9\beta\{\beta G' r_{00} - H s_0 - I r_0\} = 0, \tag{5.6}$$

where

$$F' = \frac{F - 9\beta^4}{\alpha^2},$$
$$G' = \frac{G - 3n\beta^2}{\alpha^2}.$$

The term of (5.6) which does not contain β is $b^4\alpha^2(k\alpha^2 + 3nr_{00})$. Consequently, we must have $hp(1)V$ i.e., $V = v_i y^i$ such that the above is equal to $b^4\alpha^2\beta V$. Thus

$$k\alpha^2 + 3nr_{00} = \beta V. \tag{5.7}$$

Since (5.7) is a contradiction, we have $k = 0$, and hence we get under the assumption that $n > 2$,

$$r_{00} = \frac{1}{3n}\beta V; \quad r_{ij} = \frac{1}{6n}(b_i v_j + b_j v_i). \tag{5.8}$$

Transvecting (5.8) by $b^i y^j$, we have

$$r_0 = \frac{1}{6n}(b^2 V + v_b\beta); \quad r_j = \frac{1}{6n}(b^2 v_j + v_b b_j), \tag{5.9}$$

where $v_b = v_i b^i$. Substituting $k = 0$, (5.8), (5.9) into (5.6), we have

$$\{2nVF' + 18nHs_0 + 3b^2VI\} = 3\beta\{2\beta VG' - v_bV\}. \tag{5.10}$$

The term of (5.10) which does not contain β is $b^2\alpha^2[(2n-3)\{Vb^2 - 18ns_0\}]$. Thus we must have $hp(2)V_2$, such that $b^2\alpha^2[(2n-3)\{Vb^2 - 18ns_0\}] = \beta V_2$. Hence there exist a function $h(x)$ such that

$$\begin{aligned} s_0 =& \frac{1}{18n(2n-3)}[(2n-3)vb^2 - h\beta], \\ s_j =& \frac{1}{18n(2n-3)}[(2n-3)v_j b^2 - hb_j] \end{aligned} \tag{5.11}$$

Conversely, substituting $k = 0$ and (5.8) into (5.4), we have $9B^m_m = v$, that is, B^m_m is $hp(1)$.

Next suppose that $\alpha^2 \equiv 0(mod\beta)$, that is, $n = 2$, $b^2 = 0$ and $\alpha^2 = \beta\delta$, $\delta = d_i y^i$, $d_i b^i = 2$. Since the dimension is equal to 2 and (b_i, d_i) are independent pair, we can put $v_i = f(x)b_i + g(x)d_i$ under two functions $f(x)$ and $g(x)$, and then $v_b = 2g$. Transvection of (5.11) by b^i gives $g = 0$. Hence $v_i = f(x)b_i$ and $v_b = 0$. With this (5.8) becomes

$$3nr_{00} = f(x)\beta^2; \quad 3nr_{ij} = f(x)b_i b_j. \tag{5.12}$$

From (5.9), we get $r_j = 0$, and from (5.11), we get

$$\begin{aligned} 18n(2n-3)s_0 &= -\,h(x)\beta, \\ 18n(2n-3)s_j &= -\,h(x)b_j. \end{aligned} \tag{5.13}$$

Conversely, substituting $\alpha^2 = \beta\delta$ and (5.12) into (5.4), we have $9B^m_m = f(x)\beta + k\delta$, that is, B^m_m is $hp(1)$. Hence we have

Theorem 5.3. A Finsler space with the metric $L^2 = 2\alpha\beta$ is a weakly Berwald space if and only if the following conditions holds;

(1) $\alpha^2 \not\equiv 0(\mod \beta)$ implies (5.8) and (5.11) are satisfied under $n > 2$ and $v_b = v_i b^i$.

(2) $\alpha^2 \equiv 0(\mod \beta)$ implies $n = 2$, $b^2 = 0$ and (5.12), (5.13) are satisfied, where $\alpha^2 = \beta\delta$, $\delta = d_i y^i$ and $f(x)$, $h(x)$ are functions of (x^i).

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