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Some types of trans-Sasakian manifolds

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Abstract

In this paper pseudo projectively flat and conharmonically flat trans-Sasakian manifold satisfying $R(X,Y).S = 0$ with vector ξ belonging to k-nullity distribution have been studied.

Further we have studied Legendre curves in trans-Sasakian manifold.

Key words : k-nullity distribution in trans-Sasakian manifold, Legendre curves, Levi-Civita connection, Lorentzian Sasaki spaces.
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1. Introduction

A $(2n+1)$ dimensional, $(n \geq 1)$ almost contact metric manifold M with almost contact structure (φ, ξ, η, g), where φ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric such that

$$\varphi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \ \varphi(\xi) = 0, \ \eta \ o \ \varphi = 0 \tag{1.1}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{1.2}$$

$$g(X, \varphi Y) = -g(\varphi X, Y), \qquad g(X, \xi) = \eta(X) \tag{1.3}$$

for all $X, Y \in TM$,

is called trans-Sasakian manifold [4] if and only if

$$(\nabla_X \varphi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\varphi X, Y)\xi - \eta(Y)\varphi X) \tag{1.4}$$

for some smooth functions α and β on M and ∇_X is covariant differentiation along X.

In trans-Sasakian manifold the following results hold [7]

$$\nabla_X \xi = -\alpha\varphi X + \beta(X - \eta(X)\xi) \tag{1.5}$$

$$(\nabla_X \eta)Y = -\alpha g(\varphi X, Y) + \beta g(\varphi X, \varphi Y). \tag{1.6}$$

We will use the following results of [7] in the next sections.

$$\begin{aligned} R(X,Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\varphi X - \eta(X)\varphi Y) \\ + (Y\alpha)\varphi X - (X\alpha)\varphi Y + (Y\beta)\varphi^2 X - (X\beta)\varphi^2 Y \end{aligned} \tag{1.7}$$

$$R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi\beta)(\eta(X)\xi - X) \tag{1.8}$$

$$2\alpha\beta + \xi\alpha = 0 \tag{1.9}$$

$$S(X,\xi) = (2n(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - (2n-1)X\beta - (\varphi X)\alpha \tag{1.10}$$

$$Q\xi = (2n(\alpha^2 - \beta^2) - \xi\beta)\xi - (2n-1)\text{grad }\beta + \varphi(\text{grad }\alpha) \tag{1.11}$$

where R is the curvature tensor, S is the Ricci-tensor and r is the scalar curvature. Also,

$$g(QX, Y) = S(X, Y) \tag{1.12}$$

Q being the symmetric endomorphism of the tangent space at each point corresponding to the Ricci-tensor S.

When

$$\varphi(\text{grad }\alpha) = (2n-1)\text{grad }\beta, \tag{1.13}$$

(1.10) and (1.11) reduces to

$$S(X,\xi) = 2n(\alpha^2 - \beta^2)\eta(X) \tag{1.14}$$

and

$$Q\xi = 2n(\alpha^2 - \beta^2)\xi. \tag{1.15}$$

Again a trans-Sasakian manifold is said to be locally φ-symmetric [10] if

$$\varphi^2((\nabla_W R)(X,Y)Z) = 0 \tag{1.16}$$

for all vector fields X, Y, Z, W orthogonal to ξ. The k-nullity distribution [9] of a Riemannian manifold (M, g), for a real number k, is a distribution

$$N(k) : p \to N_p(k) = [Z \in T_pM : R(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\}] \tag{1.17}$$

for all $X, Y \in T_pM$. Hence if the characteristic vector field ξ of the contact metric manifold M^{2n+1} belongs to the k-nullity distribution then we have

$$R(X,Y)\xi = k\{\eta(Y)X - \eta(X)Y\} \tag{1.18}$$

$$S(X,\xi) = 2nk\eta(X). \tag{1.19}$$

We can define k-nullity distribution in a trans-Sasakian manifold by

$$\begin{aligned} N(k) : p \to N_p(k) = [Z \in T_pM : R(X,Y)Z \\ = k[(\alpha^2 - \beta^2)(g(Y,Z)X - g(X,Z)Y) \\ + 2\alpha\beta(g(Y,Z)\varphi X - g(X,Z)\varphi Y) \\ + (Y\alpha)\varphi X - (X\alpha)\varphi Y + (Y\beta)\varphi^2 X - (X\beta)\varphi^2 Y]]. \end{aligned} \tag{1.20}$$

So when $\xi, Z \in N(k)$ we have,

$$\eta(R(X,Y)Z) = k(\alpha^2 - \beta^2)\{(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} \tag{1.21}$$

$$S(X,Z) = (2nk(\alpha^2 - \beta^2) - \xi\beta)g(X,Z) - (2n-1)X\beta - (\varphi X)\alpha \tag{1.22}$$

$$S(X,\xi) = (2nk(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - (2n-1)X\beta - (\varphi X)\alpha. \tag{1.23}$$

From (1.13) we get

$$S(X,\xi) = 2nk(\alpha^2 - \beta^2)\eta(X) \tag{1.24}$$

and

$$Q\xi = 2nk\,(\alpha^2 - \beta^2)\xi \qquad \text{when } \xi \in N(k). \tag{1.25}$$

A pseudo projective curvature tensor in a Riemannian manifold is defined [5] as

$$\begin{aligned} \tilde{P}(X,Y)Z = \; & aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] \\ & - \frac{r}{2n+1}\left[\frac{a}{2n} + b\right][g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{1.26}$$

where a, b are constants such that $a, b \neq 0$.

A conharmonic curvature tensor in a Riemannian manifold is defined as

$$\begin{aligned} H(X,Y)Z = \; & R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y \\ & + g(Y,Z)QX - g(X,Z)QY]. \end{aligned} \tag{1.27}$$

From [2] we know that, a 1-dimensional integral sub manifold in the contact subbundle is called a Legendre curve. If $\gamma(s)$ be a curve in a Riemannian manifold parametrized by the arc length then it is called Frenet curve of osculating order r if there exist orthogonal vector fields $E_1, E_2, ..., E_r$ along γ such that

$$\dot{\gamma} = E_1,\ \nabla_{\dot{\gamma}} E_1 = k_1 E_2,\ \nabla_{\dot{\gamma}} E_2 = -k_1 E_1 + k_2 E_3, ..., \nabla_{\dot{\gamma}} E_r = -k_{r-1} E_{r-1} \tag{1.28}$$

where $k_1, k_2, ..., k_{r-1}$ are positive smooth functions of s and ∇ is Levi-Civita connection of M.

Also a Frenet curve of osculating order 2 with k_1 is constant is called a circle.

In this paper we have studied Legendre curve in a trans-Sasakian manifold.

2. Pseudo projectively flat trans-Sasakian manifold with $\boldsymbol{\xi} \in N(\boldsymbol{k})$ satisfying $\varphi(\mathbf{grad}\,\boldsymbol{\alpha}) = (\mathbf{2}\boldsymbol{n} - \mathbf{1})\mathbf{grad}\,\boldsymbol{\beta}$

Let us consider a trans-Sasakian manifold M^{2n+1} with $\xi \in N(k)$. Since the manifold is pseudo projectively flat, we have, $\tilde{P}(X,Y)Z = 0$ for all X, Y and Z.

Hence from (1.26)

$$\begin{aligned} aR(X,Y)Z \ &= b[S(X,Z)Y - S(Y,Z)X] \\ &\quad - \frac{r}{2n+1}\left[\frac{a}{2n} + b\right][g(X,Z)Y - g(Y,Z)X]. \end{aligned} \tag{2.1}$$

Taking inner product on both sides of (2.1) by ξ and using (1.3) and (1.24) we obtain that

$$\begin{aligned} &\{ak(\alpha^2 - \beta^2) - \frac{r}{2n(2n+1)}[a + 2nb]\}[(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ &\quad + b[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)] = 0. \end{aligned} \tag{2.2}$$

Substituting X by ξ in (2.2) we get by virtue of (1.1), (1.3), (1.13) and (1.24) that

$$\begin{aligned} -bS(Y,Z) = &\{ak(\alpha^2 - \beta^2) - \frac{r}{2n(2n+1)}(a + 2nb)\}g(Y,Z) \\ &-\{ak(\alpha^2 - \beta^2) - \frac{r}{2n(2n+1)}(a + 2nb) + 2nbk(\alpha^2 - \beta^2)\}\eta(Y)\eta(Z). \end{aligned} \tag{2.3}$$

Putting $Y = Z = e_i$, where $\{e_i\}$ be an orthonormal basis of the tangent space at any point of the manifold, in (2.3) and taking summation over i, $1 \le i \le 2n+1$, we get

$$r = 2nk(2n+1)(\alpha^2 - \beta^2) \qquad \text{provided} \quad a \neq b. \tag{2.4}$$

Now from (2.3) on using (2.4) we have

$$\begin{aligned} -bS(Y,Z) = &\{ak(\alpha^2-\beta^2) - \frac{2nk(2n+1)(\alpha^2-\beta^2)}{2n(2n+1)}(a+2nb)\}g(Y,Z) \\ &-\{ak(\alpha^2-\beta^2) - \frac{2nk(2n+1)(\alpha^2-\beta^2)}{2n(2n+1)}(a+2nb) \\ &+2nbk(\alpha^2-\beta^2)\}\eta(Z)\eta(Y) \end{aligned} \tag{2.5}$$

i.e. $\quad S(Y,Z) = 2nk(\alpha^2-\beta^2)g(Y,Z)$

i.e. the manifold is a η-Einstein manifold.

Now using (2.5) in (2.1) we get

$$R(X,Y)Z = k(\alpha^2-\beta^2)[g(Y,Z)X - g(X,Z)Y], \text{ (Since } a \neq 0) \tag{2.6}$$

Thus we can state the theorem

Theorem 1. A pseudo projectively flat trans-Sasakian manifold with $\xi \in N(k)$, satisfying $\varphi(\text{grad } \alpha) = (2n-1)\text{grad } \beta$, is a space of constant curvature.

3. Conharmonically flat trans-Sasakian manifold with $\xi \in N(k)$ satisfying $R(X,Y).S = 0$

First we consider that the manifold is conharmonically flat.Then from (1.27) it follows that

$$R(X,Y)Z = \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \tag{3.1}$$

Next we consider in a trans Sasakian manifold with ξ belonging to k-nullity distribution satisfying $R(X,Y).S = 0$

which gives

$$S(R(X,Y)Z,W) + S(Z,R(X,Y)W) = 0. \tag{3.2}$$

Taking $Y = Z$ and using (3.1) and (3.2) we get

$$\begin{aligned} &g(Z,Z)S(QX,W) - g(X,Z)S(QZ,W) \\ &\quad +g(Z,W)S(QX,Z) - g(X,W)S(QY,Z) = 0. \end{aligned} \tag{3.3}$$

Taking $Z = \xi$ in (3.3) we have

$$\begin{aligned} &g(\xi,\xi)S(QX,W) - g(X,\xi)S(Q\xi,W) \\ &\quad +g(\xi,W)S(QX,\xi) - g(X,W)S(QY,\xi) = 0. \end{aligned} \tag{3.4}$$

Considering $\varphi(\operatorname{grad}\alpha) = (2n-1)\operatorname{grad}\beta$ we get

$$\begin{aligned} &S(QX,W) - 4n^2k^2(\alpha^2-\beta^2)^2\eta(X)\eta(W) \\ &\qquad +\eta(W)S(QX,\xi) - 4k^2n^2(\alpha^2-\beta^2)^2g(X,W) = 0. \end{aligned} \tag{3.5}$$

Let λ be the eigen value of the endomorphism Q corresponding to an eigen vector X.Then

$$QX = \lambda X. \tag{3.6}$$

Now, putting (3.6) in (3.5) and using (1.12) we have

$$\begin{aligned} &\lambda^2 g(X,W) - 4n^2k^2(\alpha^2-\beta^2)^2\eta(X)\eta(W) \\ &\qquad + 2nk(\alpha^2-\beta^2)\lambda\eta(X)\eta(W) - 4k^2n^2(\alpha^2-\beta^2)^2g(X,W) = 0. \end{aligned} \tag{3.7}$$

Putting $W = \xi$ in (3.7)and using (1.1) and (1.3)

$$[\lambda^2 - 8n^2k^2(\alpha^2-\beta^2)^2 + 2nk\lambda(\alpha^2-\beta^2)]\eta(X) = 0. \tag{3.8}$$

As $\eta(X) \neq 0$, we have

$$[\lambda^2 - 8n^2k^2(\alpha^2-\beta^2)^2 + 2nk\lambda(\alpha^2-\beta^2)] = 0 \tag{3.9}$$

i.e.

$$\lambda_1 = -4nk(\alpha^2-\beta^2) \qquad \text{and} \qquad \lambda_2 = 2nk(\alpha^2-\beta^2) \tag{3.10}$$

and

$$\lambda_1 + \lambda_2 = -2nk(\alpha^2-\beta^2). \tag{3.11}$$

Again from (3.1) we get

$$\begin{aligned} (2n-1)g(R(X,Y)Z,W) = &\, g(Y,Z)g(QX,W) - g(X,Z)g(QY,W) \\ &+ S(Y,Z)g(X,W) - S(X,Z)g(Y,W). \end{aligned} \tag{3.12}$$

Putting $X = W$ in (3.12) we have

$$\begin{aligned} (2n-1)g(R(W,Y)Z,W) = &\, g(Y,Z)g(QW,W) - g(W,Z)g(QY,W) \\ &+ S(Y,Z)g(W,W) - S(W,Z)g(Y,W). \end{aligned} \tag{3.13}$$

The sum for $1 \leq i \leq 2n+1$ of the above expression for $W = e_i$ yields

$$rg(Y,Z) = 0 \tag{3.14}$$

where r is the scalar curvature of the manifold and $\{e_i\}$ is an orthonormal basis of the tangent space of M. So

$$r = 0. \tag{3.15}$$

Since the scalar curvature is trace Q, we get

$$r = m\lambda_1 + (2n - m + 1)\lambda_2 \tag{3.16}$$

where m is a positive integer which is the multiplicity of λ_1 and $(2n - m + 1)$ is the multiplicity of λ_2.

Then

$$m(-4nk(\alpha^2 - \beta^2)) + (2n - m + 1)(2nk(\alpha^2 - \beta^2)) = 0 \tag{3.17}$$

i.e. $n = \dfrac{3m - 1}{2}$, $[2nk(\alpha^2 - \beta^2) \neq 0]$

Now if m is odd

$$n = 1, 4, 7, 10, 13, 16, 19, ...$$

and when m is even

$$n = \frac{5}{2}, \frac{11}{2}, \frac{17}{2}, \frac{23}{2}, ...$$

But as the manifold is odd dimensional, so dimensional of these type of trans-Sasakian manifold with $\xi \in N(k)$ will be $7, 13, 19, ...$etc. which is in arithmetic progression (AP) with first term 7 and common difference 6.

So we can state the theorem

Theorem 2. In a trans-Sasakian manifold M^{2n+1} $(n \geq 1)$ satisfying $\varphi(\text{grad}\,\alpha) = (2n - 1)\text{grad}\,\beta$ with ξ belonging to the k-nullity distribution,which is conharmonically flat together with $R(X, Y).S = 0$,the symmetric endomorphism Q of tangent space corresponding to S has two different non-zero eigen values $-4nk(\alpha^2 - \beta^2)$ and $2nk(\alpha^2 - \beta^2)$. Also dimensions of these manifolds are in AP having first term 7 and common difference 6.

4. Legendre curves in trans-Sasakian manifold

By definition of Legendre curve which is integral submanifold

$$\eta(\dot\gamma) = 0. \tag{4.1}$$

Differentiating $\eta(\dot\gamma)$ along $\dot\gamma$, we get

$$\nabla_{\dot\gamma}\eta(\dot\gamma) = 0$$

$$(\nabla_{\dot\gamma}\eta)(\dot\gamma) + \eta(\nabla_{\dot\gamma}\dot\gamma) = 0. \tag{4.2}$$

Now $(\nabla_{\dot\gamma}\eta)(\dot\gamma) = \nabla_{\dot\gamma}\eta(\dot\gamma) - \eta(\nabla_{\dot\gamma}\dot\gamma)$.

Using (1.3) and $(\nabla_{\dot{\gamma}} g)(\dot{\gamma}, \xi) = 0$ we have

$$(\nabla_{\dot{\gamma}} \eta)(\dot{\gamma}) = g(\nabla_{\dot{\gamma}} \xi, \dot{\gamma}) \tag{4.3}$$

Replacing (4.3) in (4.2) we have

$$g(\nabla_{\dot{\gamma}} \xi, \dot{\gamma}) + \eta(\nabla_{\dot{\gamma}} \dot{\gamma}) = 0. \tag{4.4}$$

Using (1.5) and (4.1) we get

$$g(-\alpha\varphi\dot{\gamma}, \dot{\gamma}) + g(\beta\dot{\gamma}, \dot{\gamma}) + \eta(\nabla_{\dot{\gamma}} \dot{\gamma}) = 0. \tag{4.5}$$

As $E_1 = \dot{\gamma}$ is orthonormal basis $g(\dot{\gamma}, \dot{\gamma}) = 1$.

By (1.3) we have

$$g(\varphi\dot{\gamma}, \dot{\gamma}) = 0.$$

Using (1.3) and the fact $g(\dot{\gamma}, \dot{\gamma}) = 1$ we obtain

$$\beta + \eta(\nabla_{\dot{\gamma}} \dot{\gamma}) = 0. \tag{4.6}$$

If $\nabla_{\dot{\gamma}} \dot{\gamma}$ is parallel to ξ, then

$$\nabla_{\dot{\gamma}} \dot{\gamma} = -\beta\xi. \tag{4.7}$$

Also by Frenet formula (1.28) we have

$$\nabla_{\dot{\gamma}} \dot{\gamma} = k_1 E_2 \tag{4.8}$$

where $E_1 = \dot{\gamma}$.

From (4.7) and (4.8)

$$E_2 = -\frac{\beta}{k_1}\xi. \tag{4.9}$$

Now, differentiating along $\dot{\gamma}$ to (4.9) and using (1.5) we get

$$\nabla_{\dot{\gamma}} E_2 = \frac{\alpha\beta}{k_1}\varphi\dot{\gamma} - \frac{\beta^2}{k_1}\dot{\gamma}. \tag{4.10}$$

Comparing with second equation of (1.28) we get curvature $R = \frac{\beta^2}{k_1}$ and torsion $\tau = \frac{\alpha\beta}{k_1}$ and the ratio of curvature and torsion becomes $\frac{\beta}{\alpha}$.

Thus we can state

Theorem 3. In a trans -Sasakian manifold a Legendre curve parametrized by arc length has curvature and torsion $\frac{\beta^2}{k_1}$ and $\frac{\alpha\beta}{k_1}$ provided $\nabla_{\dot{\gamma}} \dot{\gamma}$ is parallel to ξ, and ratio of curvature to torsion is $\frac{\beta}{\alpha}$.

References

1. Prasad, B. : A pseudo projective curvature tensor on a Riemannian manifold, Bull. Cal. Math. Soc., 94(3) (2002), 163-166.
2. Baikoussis, C. and Blair, D. E. : On Legendre curves in contact 3-manifolds, Geom. Dedicata, 49 (1994), 135-142.
3. Mihai, I. and Shahid, M. : Generic submanifolds of a trans-Sasakian manifold, Publ. Math. Debrecen, 47 3-4 (1995), 209-218.
4. Oubina, J. A. : New classes of almost contact metric structures, Publ. Math. Debrecen, 32 (1985), 187-195.
5. Tarafdar, M., Bhattacharyya, A. and Debnath, D. : A type of pseudo projective φ-recurrent trans-Sasakian manifold, Analele Ştiinţifice Ale Universităţii "AL.I.CUZA" IAŞI,Tomul LII,s.I,Matematică, 2006, f.2. 417-422.
6. Belkhelfa, M., Hirică, I. E., Rosca, R. and Verstraelen, L. : On Legendre curves in Riemannian and Lorentzian Sasaki spaces, Soochow J. Math., 28 no. 1 (2002), 81-91.
7. Tripathi, M. M. and De, U. C. : Ricci tensor in 3-dimensional trans-Sasakian manifolds, KYUNGPOOK. Math. J., 43(2003), 1-9.
8. Tripathi, M. M. : A note on certein Legendre curves in a Kenmotsu manifold, Ganita, 51 no.1 (2000), 57-58.
9. Tanno, S. : Ricci curvatures of contact Riemannian manifolds, Tohoku Math. J., 40 (1988), 441-448.
10. Takahashi, T. : Sasakian φ-symmetric spaces, Tohoku Math. J., 29 (1977), 91-113.